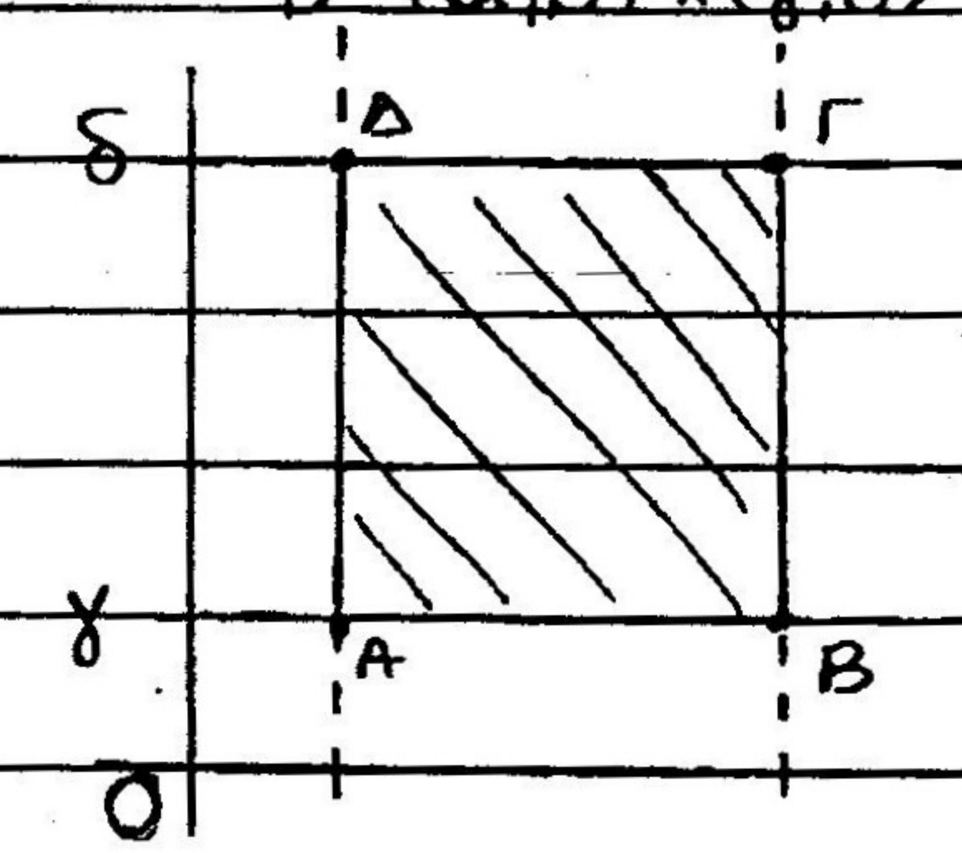
Υπολογισμός διπλού ολοκληρώματος.

① $z=f(x,y)\mid D\subset\mathbb{R}^2 \qquad f(x,y)>0\mid D$

$D=(\alpha,\beta)\times(\gamma,\delta)$

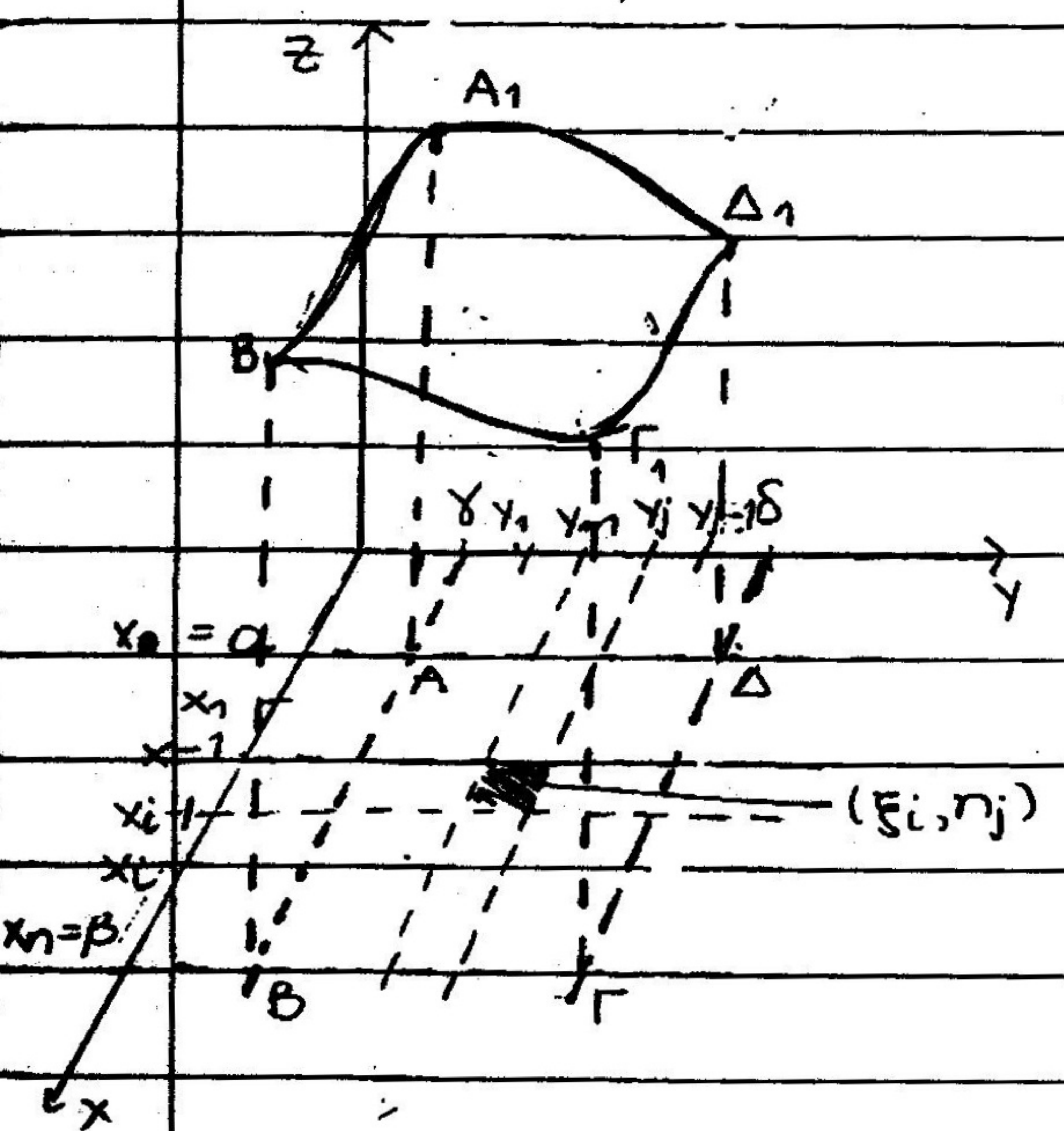


$$\iint_D f(x,y)\,dE$$

$$\lim_{\|P\|\to 0}\left\{\sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_i,n_j)E(D_{ij})\right\}=$$

$$=\lim\left\{\sum_{i=1}^{n}\underbrace{\left[\sum_{j=1}^{m} f(x_i,y_j)\Delta y_j\right]}_{\int_\gamma^\delta f(x_i,y)dy}\Delta x_i\right\}$$

$$\int_\alpha^\beta\left\{\int_\gamma^\delta f(x_i,y)dy\right\}dx$$

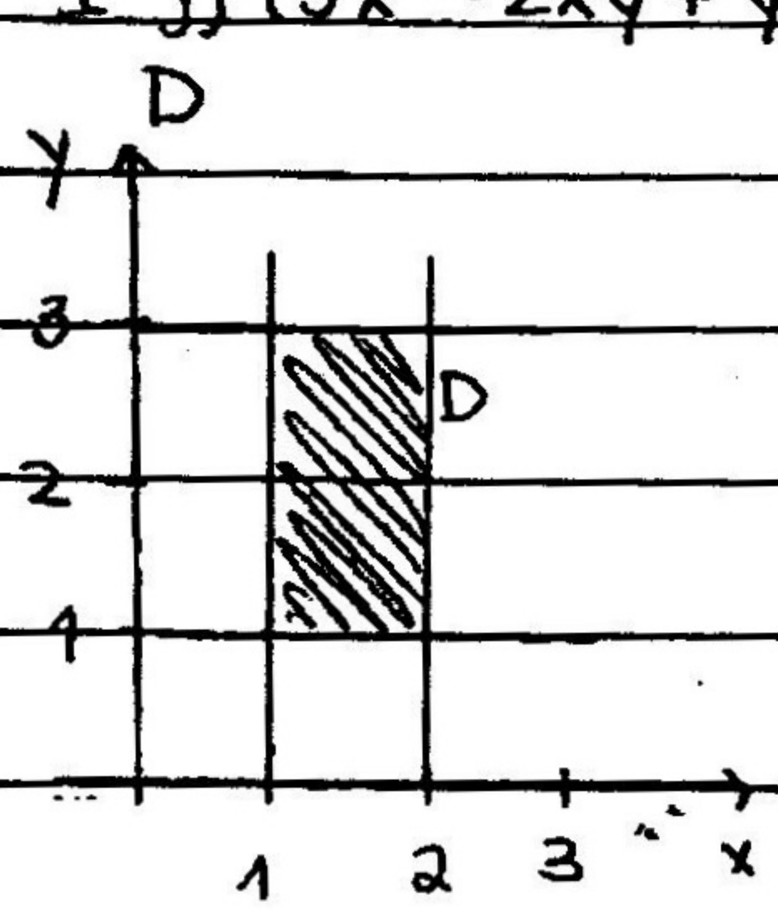


εσωτερική ολοκλήρωση → μία τομή του στερεού και βρίσκω το εμβαδόν της τομής

πάντα σχήμα τόπου ολοκλήρωσης!

Π.Χ.:

$$I\iint_D(3x^2-2xy+y^2)\,dxdy \qquad D=\begin{Bmatrix}1\le x\le 2\\ 1\le y\le 3\end{Bmatrix}$$



$$I=\int_1^2\left\{\int_1^3(3x^2-2xy+y^2)dy\right\}dx$$

$$I=\int_1^2\left\{\left[3x^2y-xy^2+{}^{y}/_{3}\right]_{y=1}^{y=3}\right\}dx$$

$$I=\int_1^2\left[9x^2-9x+9-(3x^2\quad x+{}^{1}/_{3})\right]dx$$

$$I=\int_1^2(6x^2-8x+{}^{26}/_{3})\,dx=\left[2x^3-4x^2+{}^{26}/_{3}\,x\right]_1^2=\dots$$

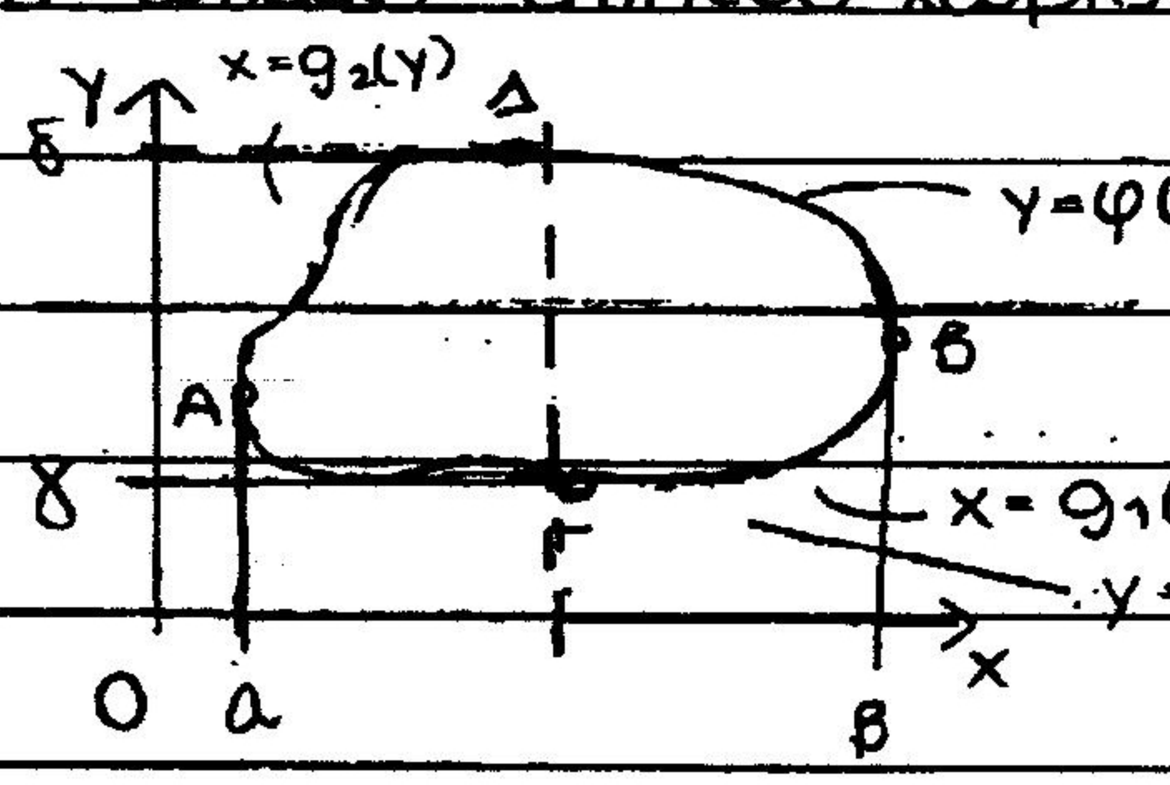
* Αλλάζοντας την σειρά ολοκλήρωσης:

$$I=\int_1^3\left\{\int_1^2(3x^2-2xy+y^2)dx\right\}dy=\int_1^3\left[x^3-x^2y+xy^2\right]_{x=1}^{x=2}dy=$$

$$I=\int_1^3\left[8-4y+2y^2-(1-y+y^2)\right]dy=\int_1^3(y^2-3y+7)\,dy=\left[\frac{y^3}{3}-\frac{3y^2}{2}+7y\right]_1^3=\dots$$

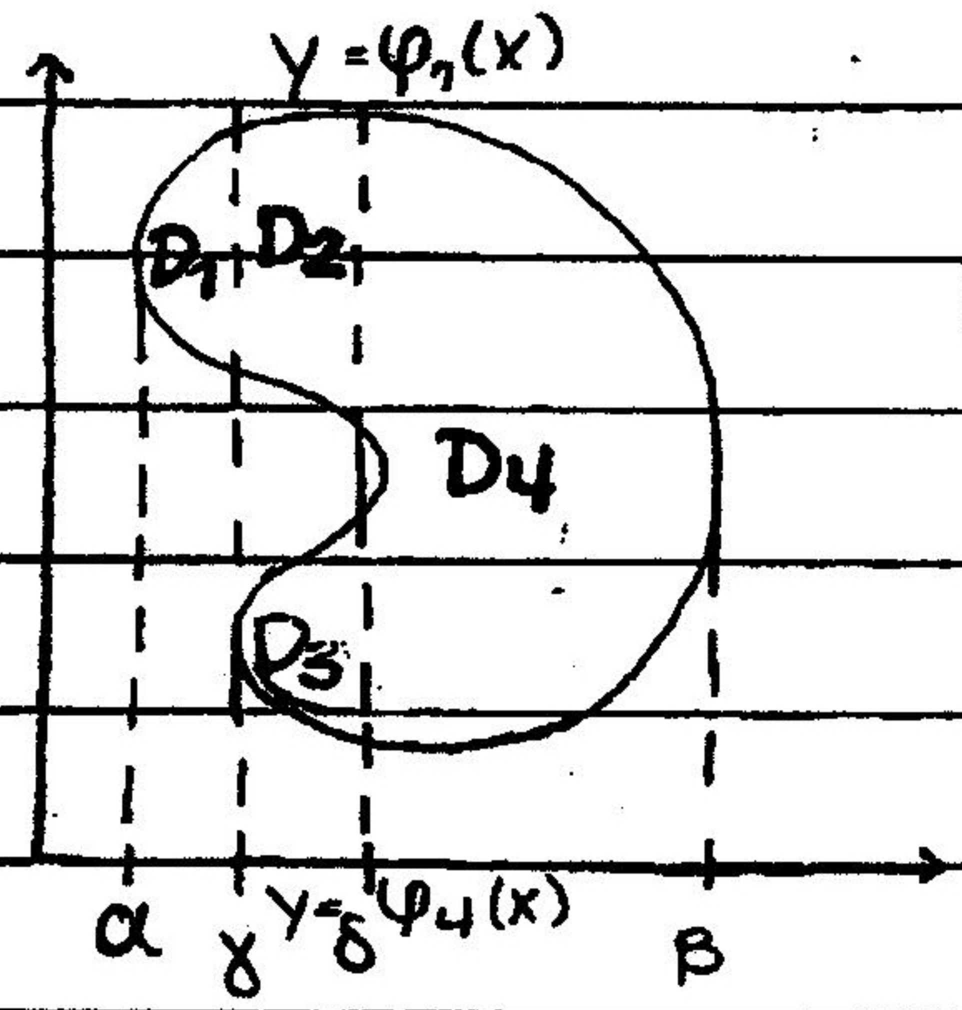
② $z=f(x,y) \mid D\subset\mathbb{R}^2 \quad f(x,y)>0 \quad \forall (x,y)\in D$

D = τυχαίο επίπεδο χωρίο του $\mathbb{R}^2$

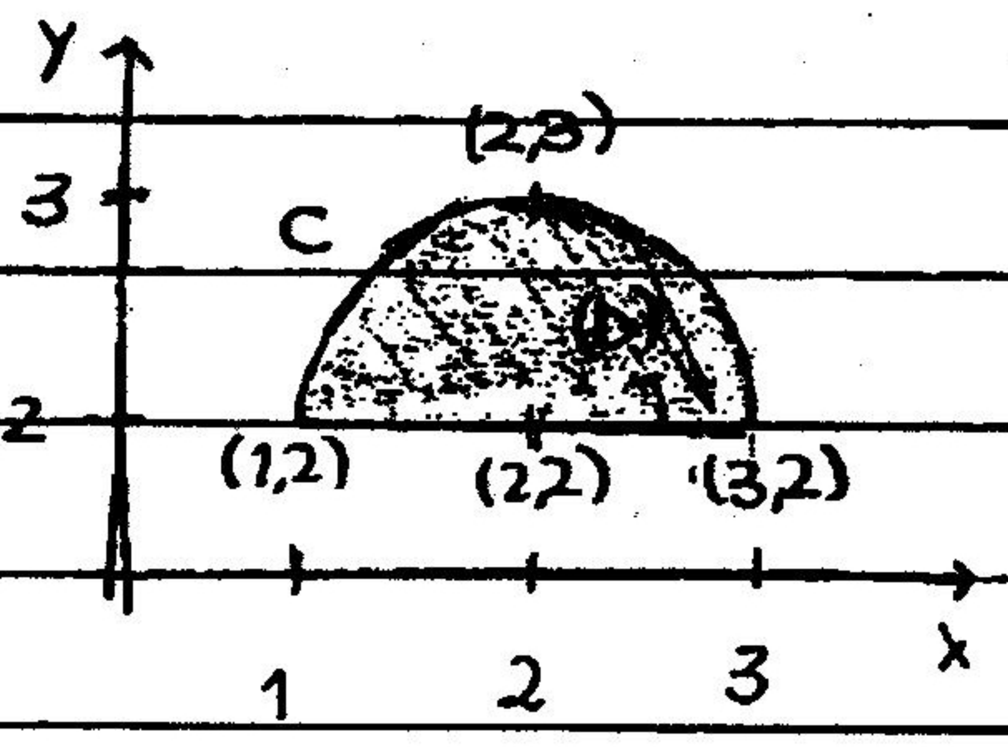


$$\iint_D f(x,y)\,dxdy = \int_\alpha^\beta \left\{ \int_{\sigma(x)}^{\varphi(x)} f(x,y)\,dy \right\} dx$$

$$\iint_D f(x,y)\,dydx = \int_\gamma^\delta \left\{ \int_{g_2(y)}^{g_1(y)} f(x,y)\,dy \right\} dy$$



$$\iint_D f(x,y)\,dxdy = \int \left\{ \int f(x,y)\,dy \right\} dx =$$
$$= \iint_{D_1} f dxdy + \iint_{D_2} f dxdy + \iint_{D_3} f dxdy + \iint_{D} f dxdy$$

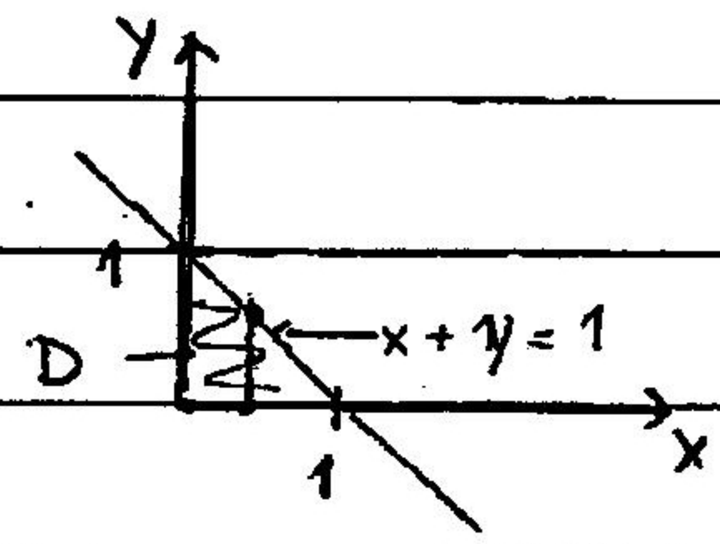


$C: (x-2)^2+(y-2)^2=1$

$(y-2)^2 = 1-(x-2)^2 \quad y = 2\pm\sqrt{1-(x-2)^2}$

$$\iint f dxdy = \int_1^{3} \left\{ \int_2^{2\pm\sqrt{1-(x-2)^2}} f\,dy \right\} dx$$

$I = \iint_D (x^2-y^2)\,dxdy \qquad D = \begin{Bmatrix} x=0 \\ y=0 \\ x+y=1 \end{Bmatrix}$



① $I = \int_0^1 \left\{ \int (x^2-y^2)\,dy \right\} dx =$

$= \int_0^1 \left[x^2y - \frac{y^3}{2} \right]_{y=0}^{y=1-x} dx = \int_0^1 \left[x^2(1-x) - \frac{(1-x)^3}{3} \right] dx$

$I = \int_0^1 \left\{ \int_0^{1-y} (x^2-y^2)\,dx \right\} dy = \int_0^1 \left[\frac{x^3}{3} - xy^2 \right]_{x=0}^{1-y} dy =$

$= \int_0^1 \left[\frac{(1-y)^3}{3} - (1-y)y^2 \right] dy = 0$

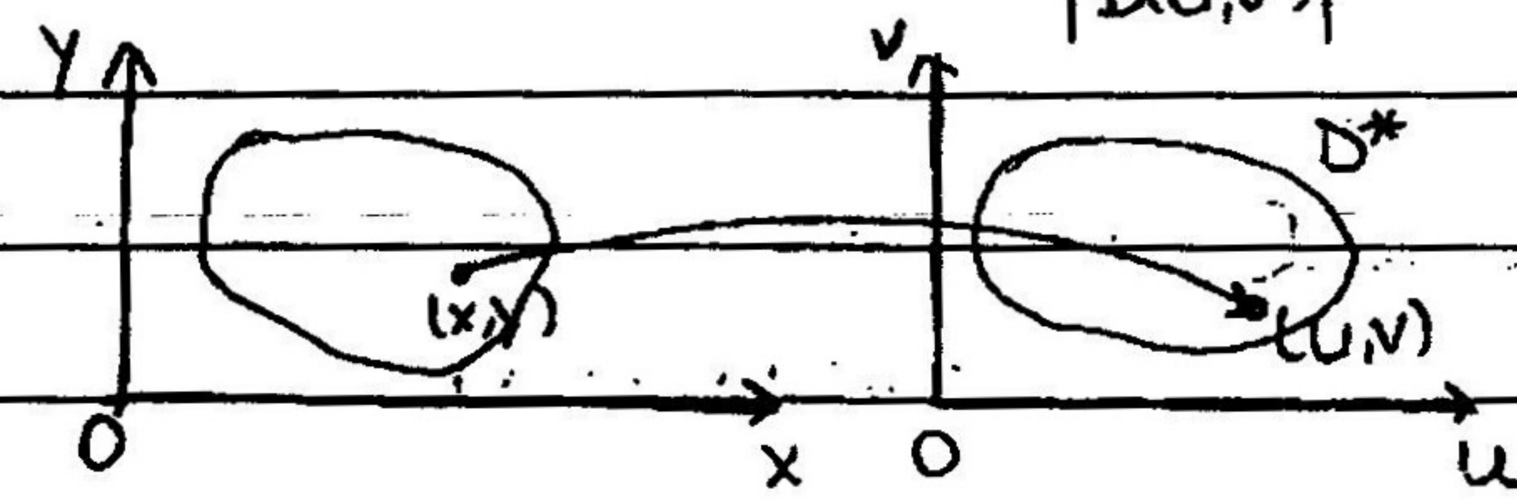
*Αν προσπαθώ να υπολογίσω ένα διπλό ολοκλήρωμα και δεν βγαίνει, πρώτα θα δοκιμάσω να αλλάξω σειρά ολοκλήρωσης και μετά σύστημα συντεταγμένων.

Αλλαγή Μεταβλητών σε Διπλό Ολοκλήρωμα.

$$I = \iint_D f(x,y)\,dx\,dy \quad \Big\| \quad I = \iint_{D^*} f(\varphi_1(u,v), \varphi_2(u,v)) \left| \frac{D(x,y)}{D(u,v)} \right| du\,dv$$

$x = \varphi_1(u,v)$

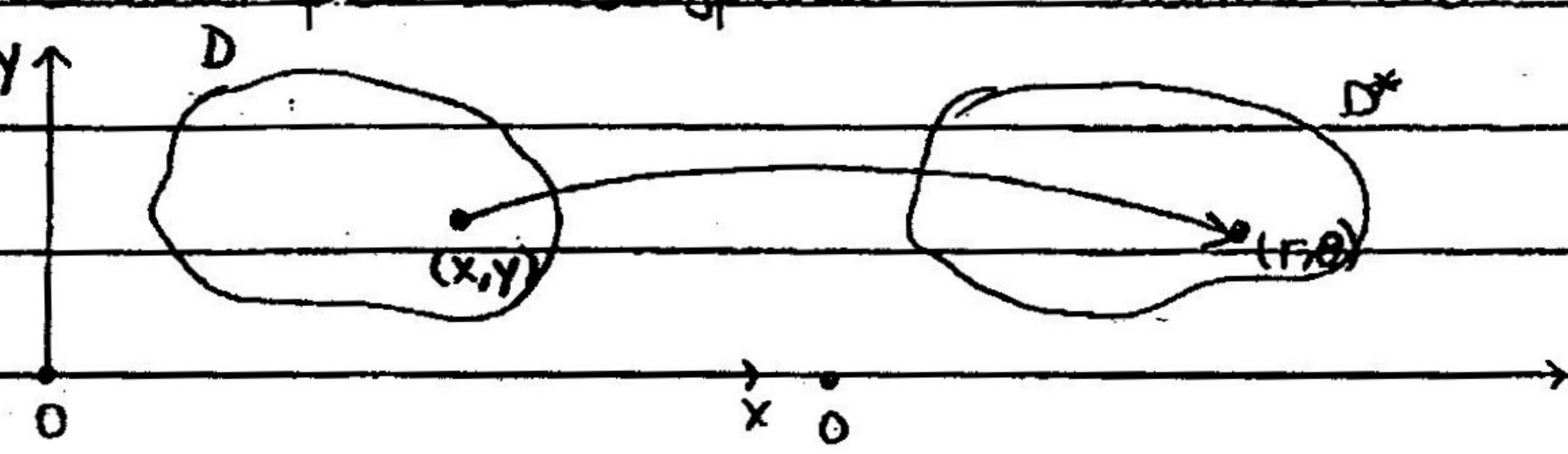
$y = \varphi_2(u,v)$



καρτεσιανό σύστημα συντεταγμένων ⟶ πολικό σ.σ

$x = r\cos\theta$

$y = r\sin\theta$



$$\frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Π.χ:

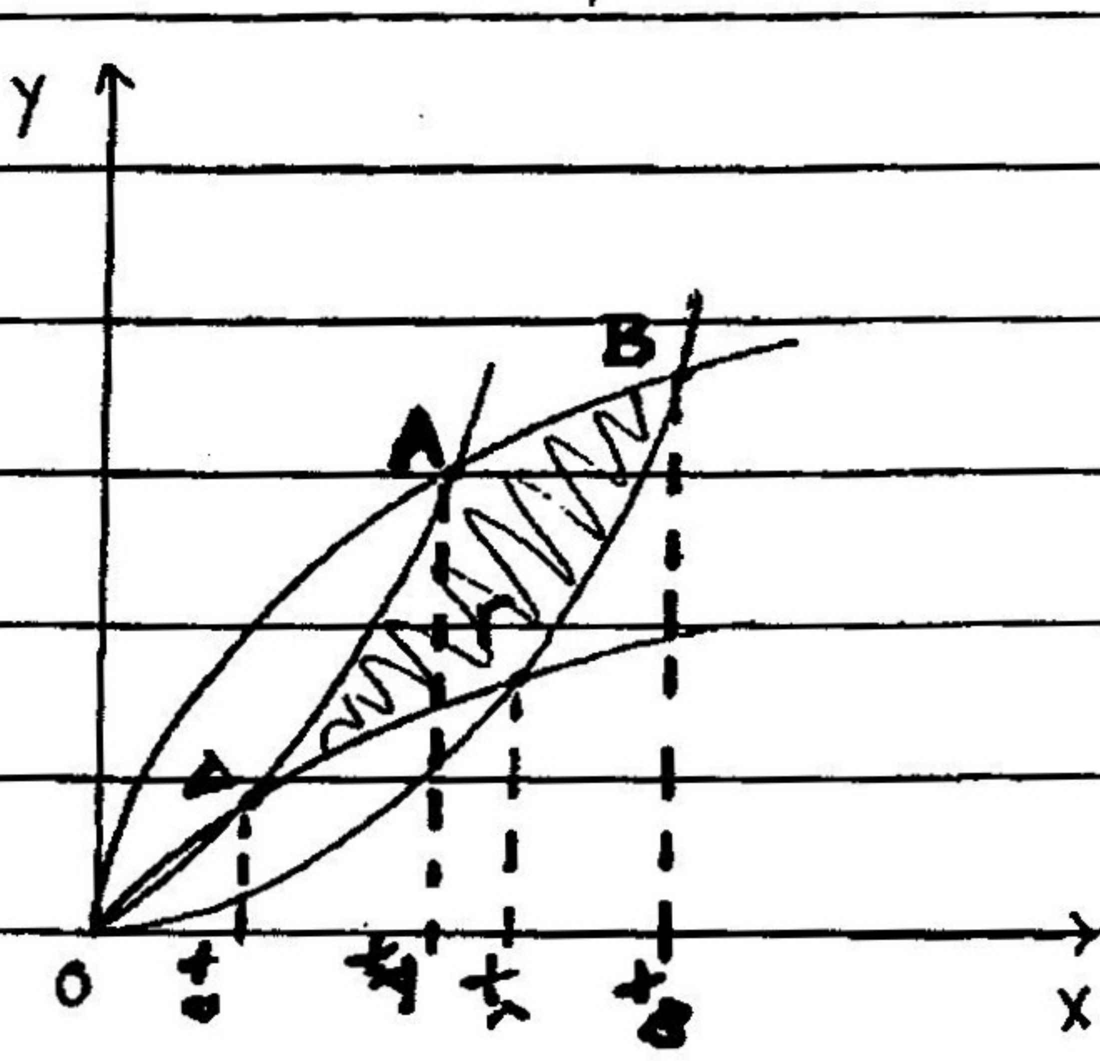
1. $I = \iint_D xy\,dx\,dy$

$D:\ x = y^2 \longrightarrow y^2/x = 1$

$y^2 = 2x \longrightarrow y^2/x = 2$

$x^2 = y \longrightarrow x^2/y = 1$

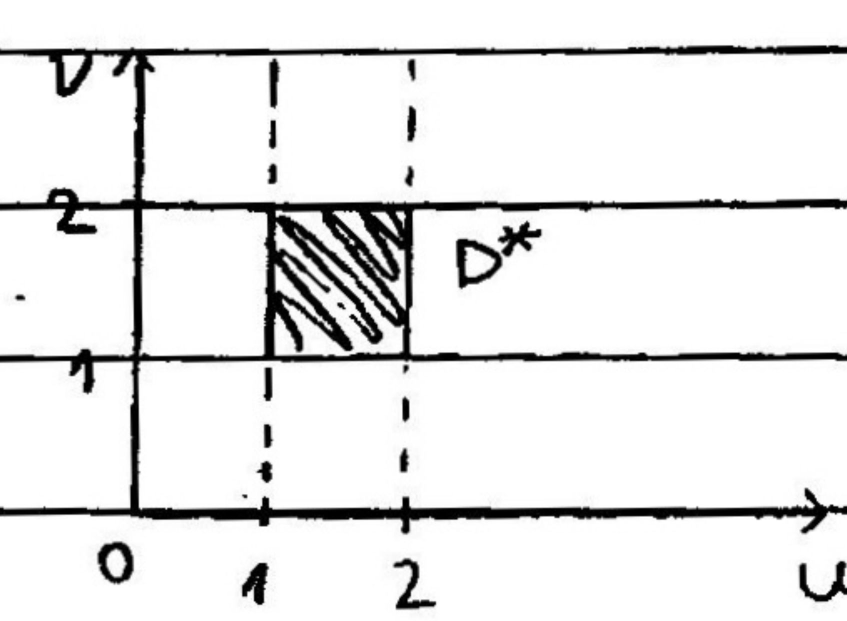
$x^2 = 2y \longrightarrow x^2/y = 2$



$u = x^2/y \quad \Big| \quad u_x = -y^2/x^2, \quad u_y = 2y/x,$

$v = y^2/x \quad \Big| \quad v_x = 2x/y, \quad v_y = -x^2/y^2$

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 3$$



$$\left| \frac{D(x,y)}{D(u,v)} \right| = \frac{1}{\left| \frac{D(u,v)}{D(x,y)} \right|} = \frac{1}{3} \qquad I = \iint_D xy\,dx\,dy = \iint_{D^*} uv\,\frac{1}{3}\,du\,dv$$

$$= \frac{1}{3} \int_1^2 u\,du \int_1^2 v\,dv = \frac{1}{3} \left[\frac{u^2}{2} \right]_1^2 \left[\frac{v^2}{2} \right]_1^2 = \ldots$$

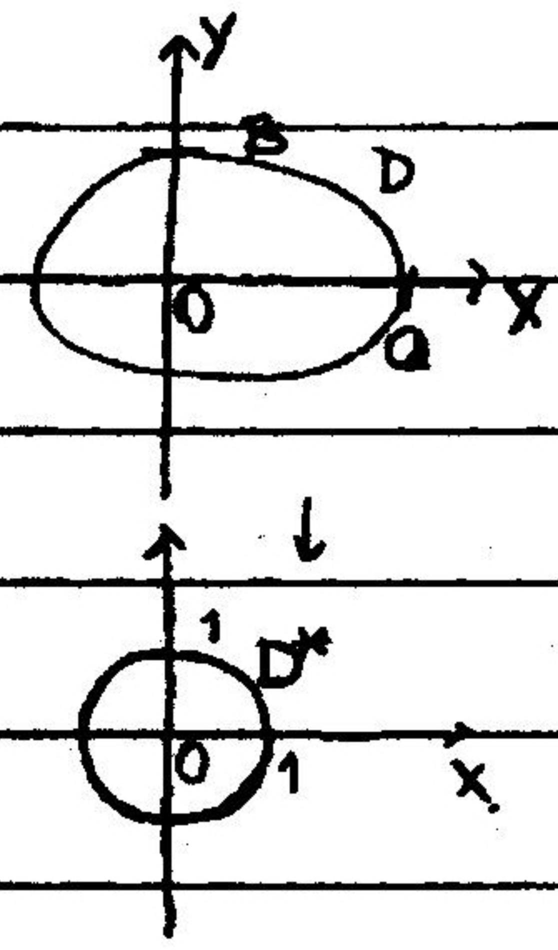
Παρατήρηση:

Αν ο τρόπος ολοκλήρωσης ενός διπλού ολοκληρώματος είναι κύκλος ή έλλειψη, ή τμήμα κύκλου, ή τμήμα έλλειψης, τότε για ακονομία πράξεων χρησιμοποιούμε πολικό σύστημα συντεταγμένων.

συγκεκριμένα: α) όταν ο τόπος ολοκλήρωσης είναι κύκλος ή τμήμα κύκλου, θέτουμε:

$x = r\cos\theta$ και $\frac{D(x,y)}{D(r,\theta)} = r$

$y = r\sin\theta$

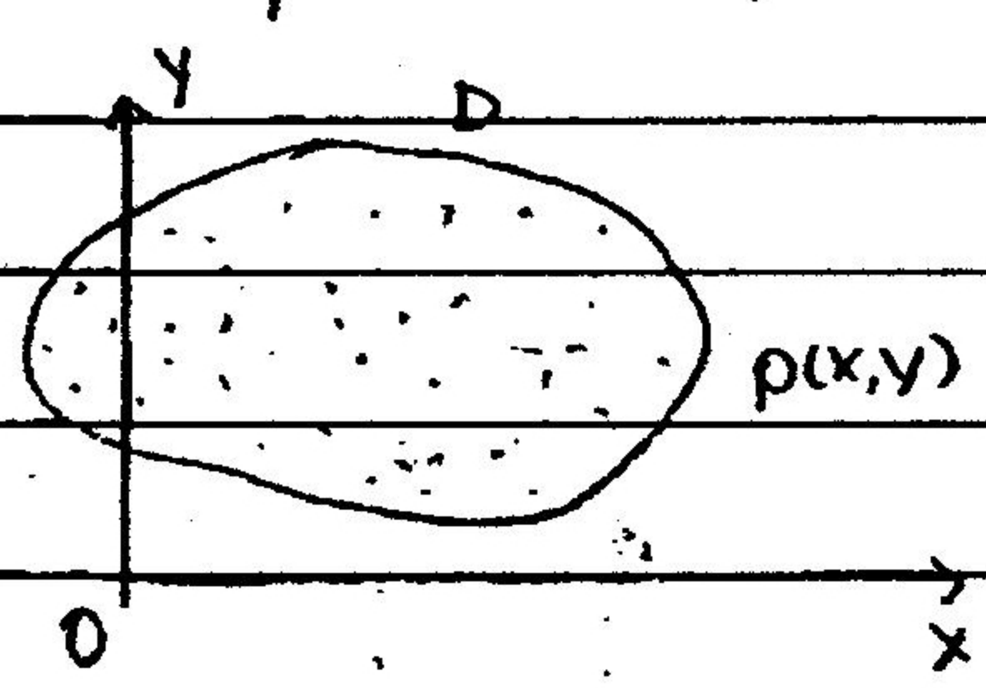


β) όταν ο τόπος ολοκλήρωσης είναι έλλειψη ή τμήμα έλλειψης με εξίσωση: $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$ θέτουμε:

$x = \alpha r\cos\theta$ και $\frac{D(x,y)}{D(r,\theta)} = \alpha\beta r$

$y = \beta r\sin\theta$

Στην περίπτωση αυτή στο πολικό σ.σ., η έλλειψη μετατρέπεται σε κύκλο.



$M = \iint_D \rho(x,y)\,dx\,dy$ κέντρο μάζας $K(x_K, y_K)$

$x_K = \frac{m_x}{M}$, $y_K = \frac{m_y}{M}$

$m_x = \iint_D x\rho\,dx\,dy$

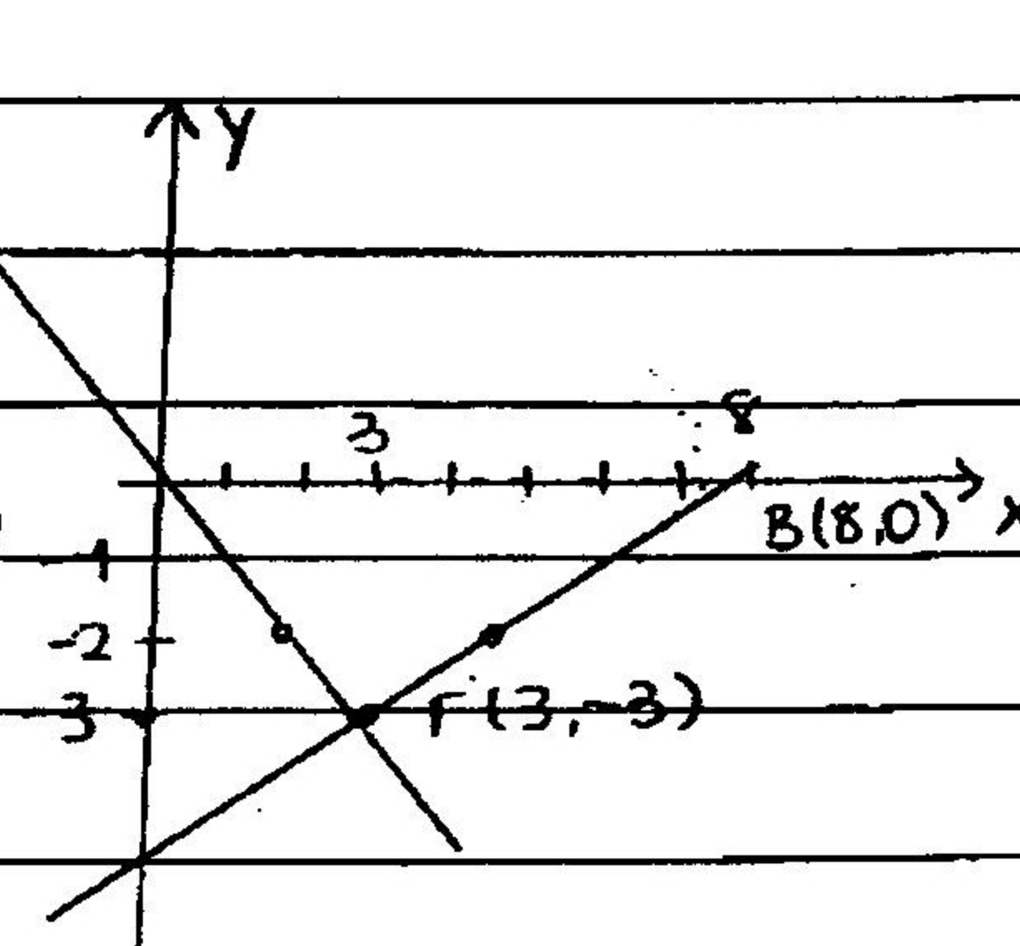
ροπές αδράνειας:

$m_y = \iint_D y\rho\,dx\,dy$ $M_x = \iint_D y^2\rho\,dx\,dy$ | $M_y = x^2\rho\,dx\,dy$

π.χ.:

2ο $\varphi(x,y) = y$

D: $y=0$ $y=-x$ $3x-5y=24$

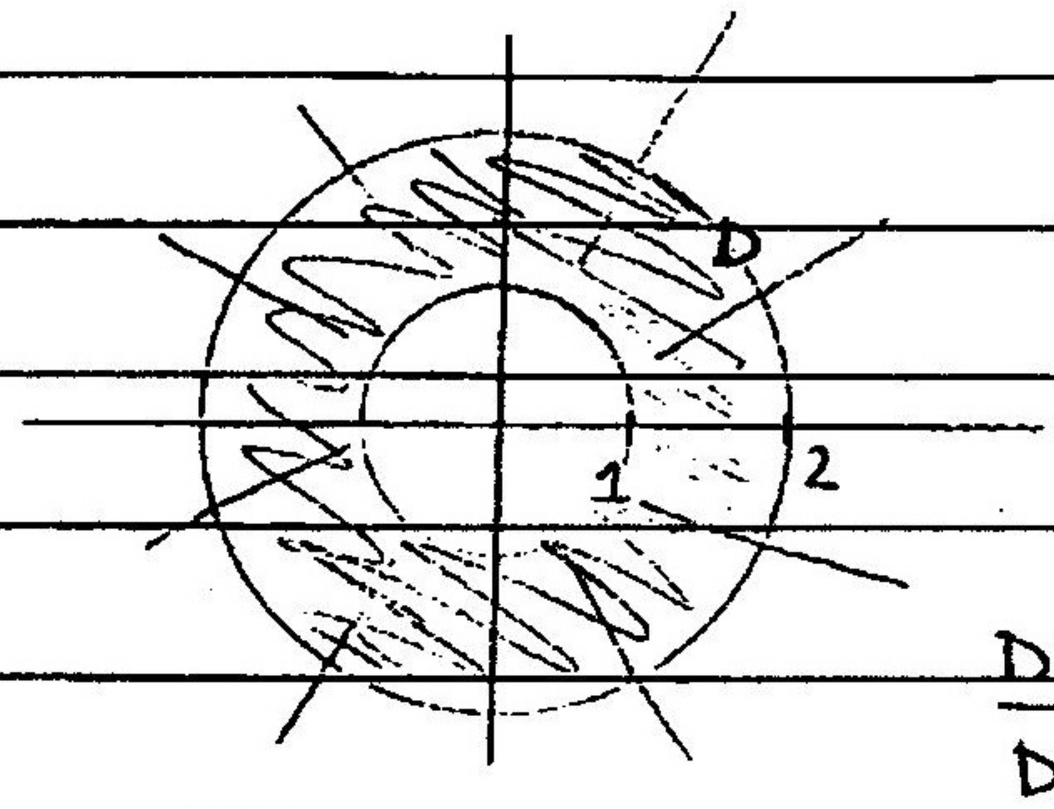


$$\iint_D y\,dx\,dy = \int_{-3}^{0}\left\{\int_{-y}^{\frac{5y+24}{3}} y\,dx\right\} = \int_{-3}^{0} y[x]_{-y}^{\frac{5y+24}{3}}\,dy$$

$$= \int_{-3}^{0} y\left[\frac{5y}{3} + 8 + y\right]dy = \ldots$$

π.χ.:

3. $\iint_D xy\,dx\,dy \qquad D:\ x^2+y^2 \geq 1,\quad x^2+y^2 \leq 4$



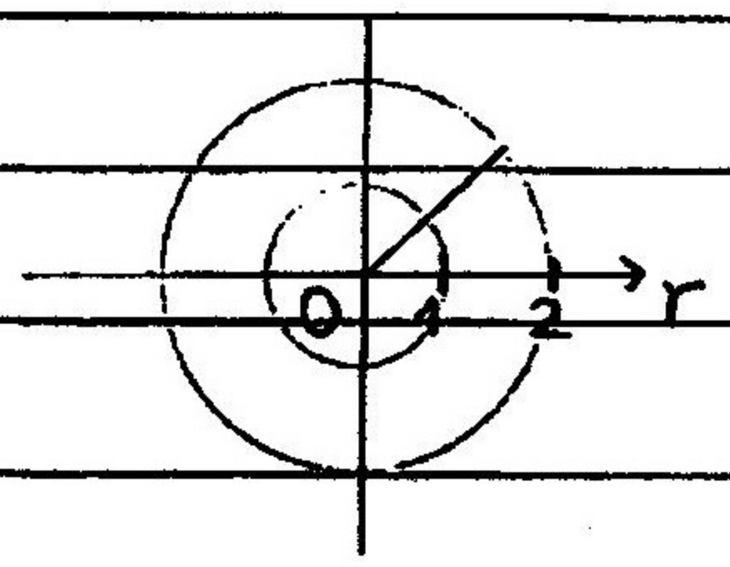
$x = r\cos\vartheta$

$y = r\sin\vartheta$

$\frac{D(x,y)}{D(r,\theta)} = r$

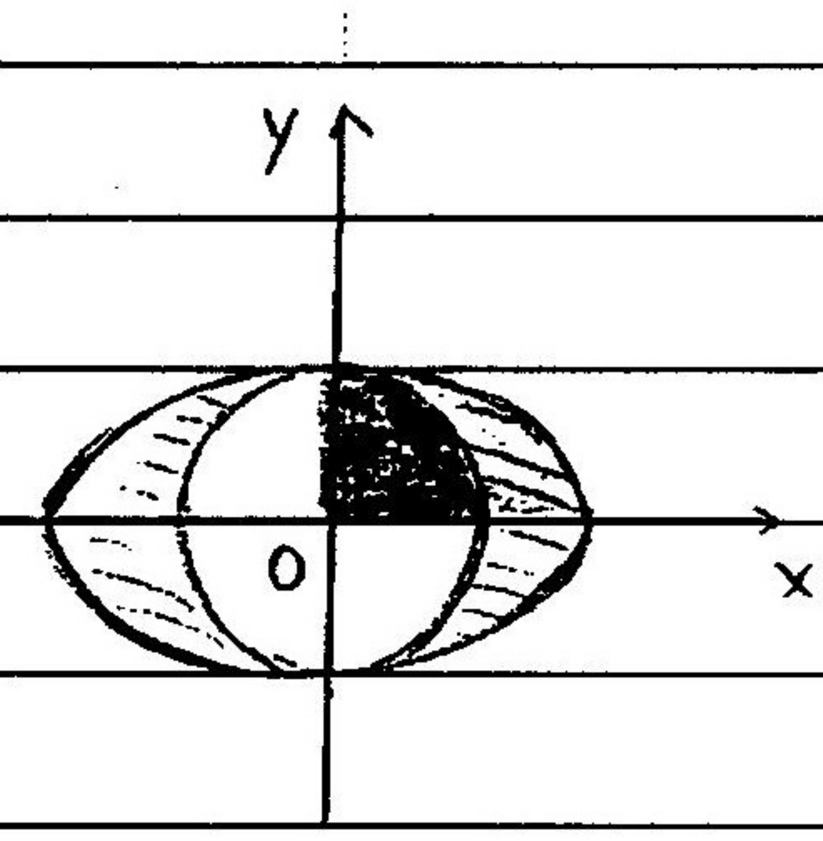
$$\iint xy\,dx\,dx = \iint_{D^*} r^2\cos\theta\sin\theta\, r\,dr\,d\theta$$

$$= \int_1^2 r^3 dr \int_0^{2\pi} \cos\theta\sin\theta\, d\theta - d(\cos\theta) = \left[\frac{r^4}{4}\right]_1^2 \left[-\frac{\cos^2\theta}{2}\right]_0^{2\pi} = 0$$



ασκήσεις:

4.(iv) $\quad I = \iint_T dx\,dy \qquad T: \begin{cases} x^2+y^2=16 \\ 16x^2+25y^2=400 \end{cases}$



$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$I = 4\iint_{D_1} dx\,dy = \underbrace{4\iint_{D_1(\text{ελλ.})} dx\,dy}_{I_1} - \underbrace{4\iint_{D_1(\text{κύκλου})} dx\,dy}_{I_2}$$

$I_2 \rightarrow$ $x = r\cos\vartheta$

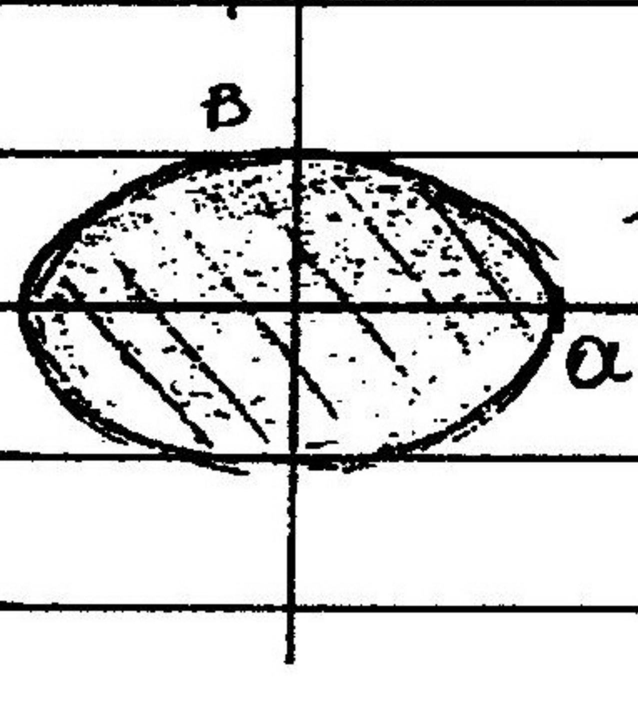
$y = r\sin\vartheta$

$\frac{D(x,y)}{D(r,\theta)} = r$

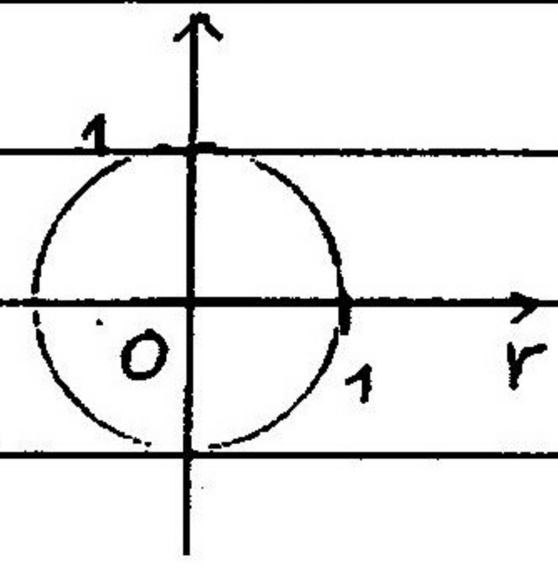
$$I = \iint_{D_1(\text{ελλ})} r\,dr\,d\theta = \int_0^{\pi/2}\left(\int_0^4 r\,dr\right)d\theta =$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^4 d\vartheta = 8\,\frac{\pi}{2} = 4\pi$$

$I_1 \rightarrow$



$\xrightarrow{\text{πολικό σσ}}$



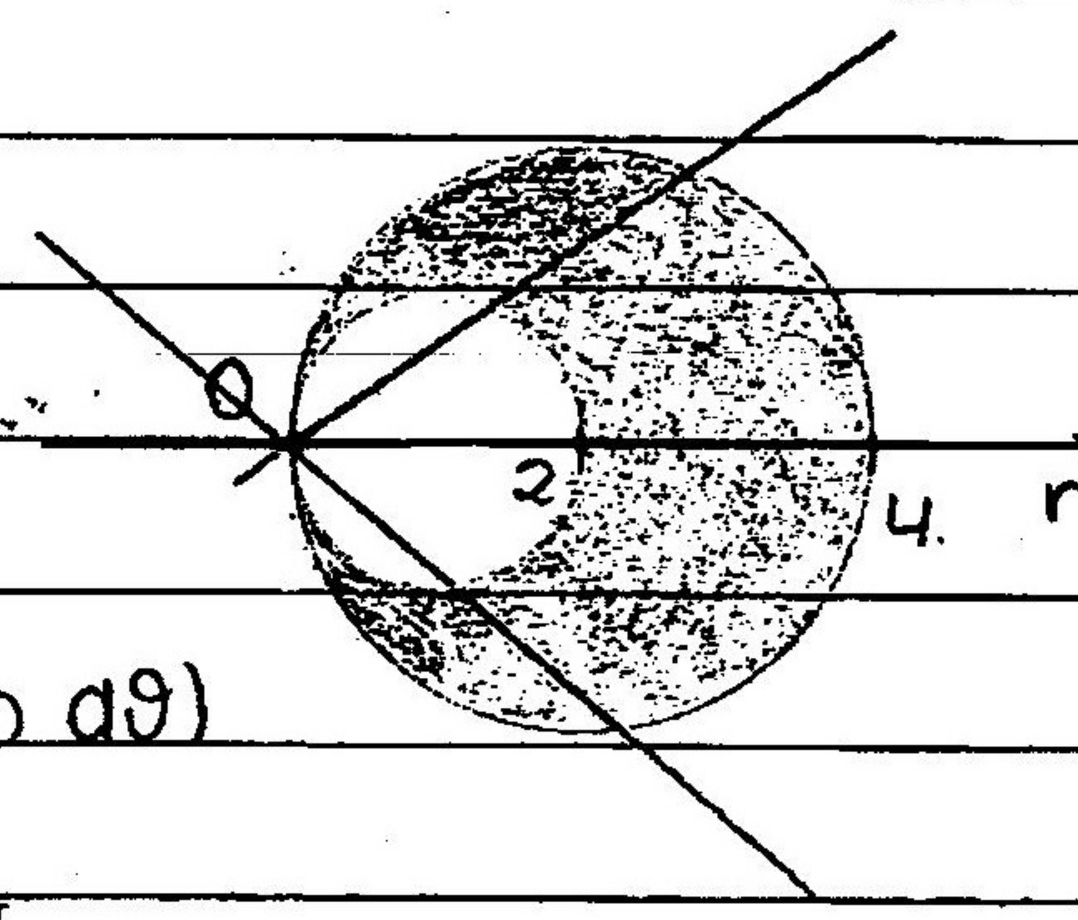
$x = \alpha r\cos\vartheta$

$y = \beta r\sin\vartheta$

$\frac{D(x,y)}{D(r,\theta)} = \alpha\beta r$

$$I_1 = \iint_{D^*_{(Ελλ)}} 20\, r\,dr\,d\theta = 20\int_0^{\pi/2}\left\{\int_0^1 r\,dr\right\} d\theta = 20\,\frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^1 = 5\pi$$

4.(v) $I = \iint_T r\,dr\,d\theta$



$T: \begin{cases} r = 2\cos\theta \\ r = 4\cos\theta \end{cases}$ (πάντα στην εξωτερική ολοκλήρωση θα έχουμε το $d\theta$)

$$I = \int_0^{\pi}\left\{\int_{2\cos\theta}^{4\cos\theta} r\,dr\right\} d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_{2\cos\theta}^{4\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi}(16\cos^2\theta - 4\cos^2\theta)\,d\theta =$$

$$= 6\int_0^{\pi}\cos^2\theta\,d\theta = \int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta = \dots = 3\pi.$$

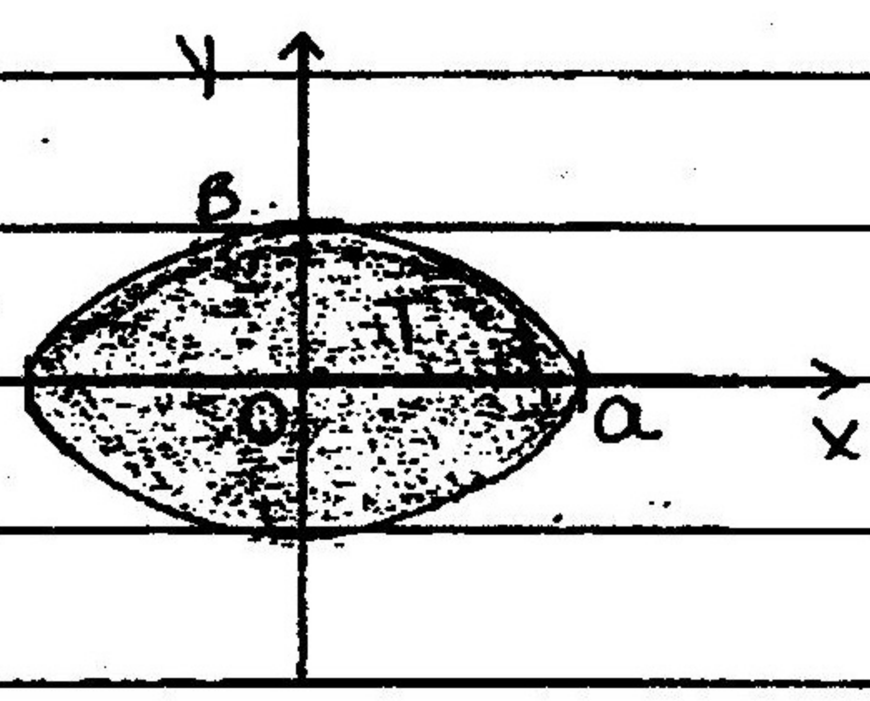
10(i) $I = \iint_T e^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{\beta^2}}}\,dx\,dy$

$x = a r\cos\theta$

$y = \beta r \sin\theta$

$\frac{D(x,y)}{D(r,\theta)} = a\beta r$

$T = \left\{(x,y) \in \mathbb{R}^2 \,/\, \frac{x^2}{a^2} + \frac{y^2}{\beta^2} \leq 1\right\}$



$$I = \iint_{D^*} e^{\sqrt{r^2}}\, a\beta r\,dr\,d\theta = a\beta\iint_{D^*} e^r r\,dr\,d\theta = a\beta\int_0^{2\pi} d\theta\int_0^1 e^r r\,dr =$$

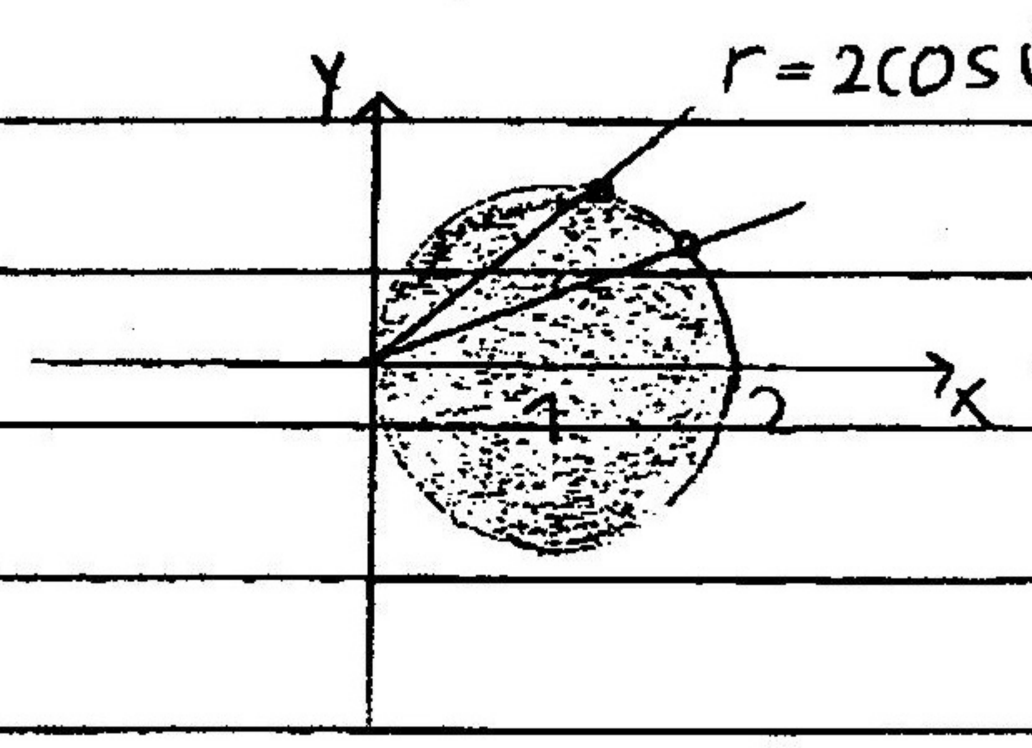
$$= 2\pi a\beta$$

10(ii) $I = \iint_T \sqrt{x^2+y^2}\,dx\,dy$

$T = \left\{(x,y) \in \mathbb{R}^2 \,/\, \underbrace{x^2+y^2 \leq 2x}\right\}$

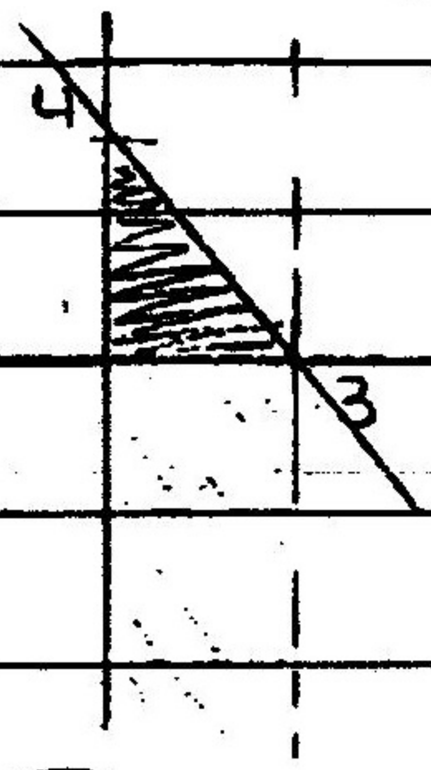
$r^{\cancel{2}} = 2\cancel{r}\cos\theta$

$r = 2\cos\theta$



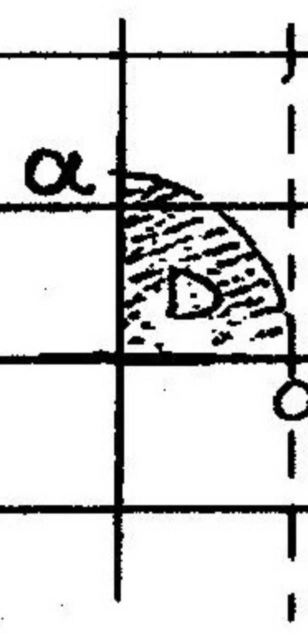
$$I = \iint_{D^*} \sqrt{r^2}\, r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left\{\int_0^{2\cos\theta} r^2\,dr\right\} d\theta$$

11(i) $\int_0^3 dx \left(\int_0^{\frac{4(3-x)}{3}} x^2 dy \right) = \int_0^3 x^2 \left[y \right]_0^{\frac{4(3-x)}{3}} dx = \int_0^3 x^2 \cdot \frac{4(3-x)}{3} dx$



$= 4/3 \int_0^3 (3x^2 - x^3) dx = \ldots = 9$

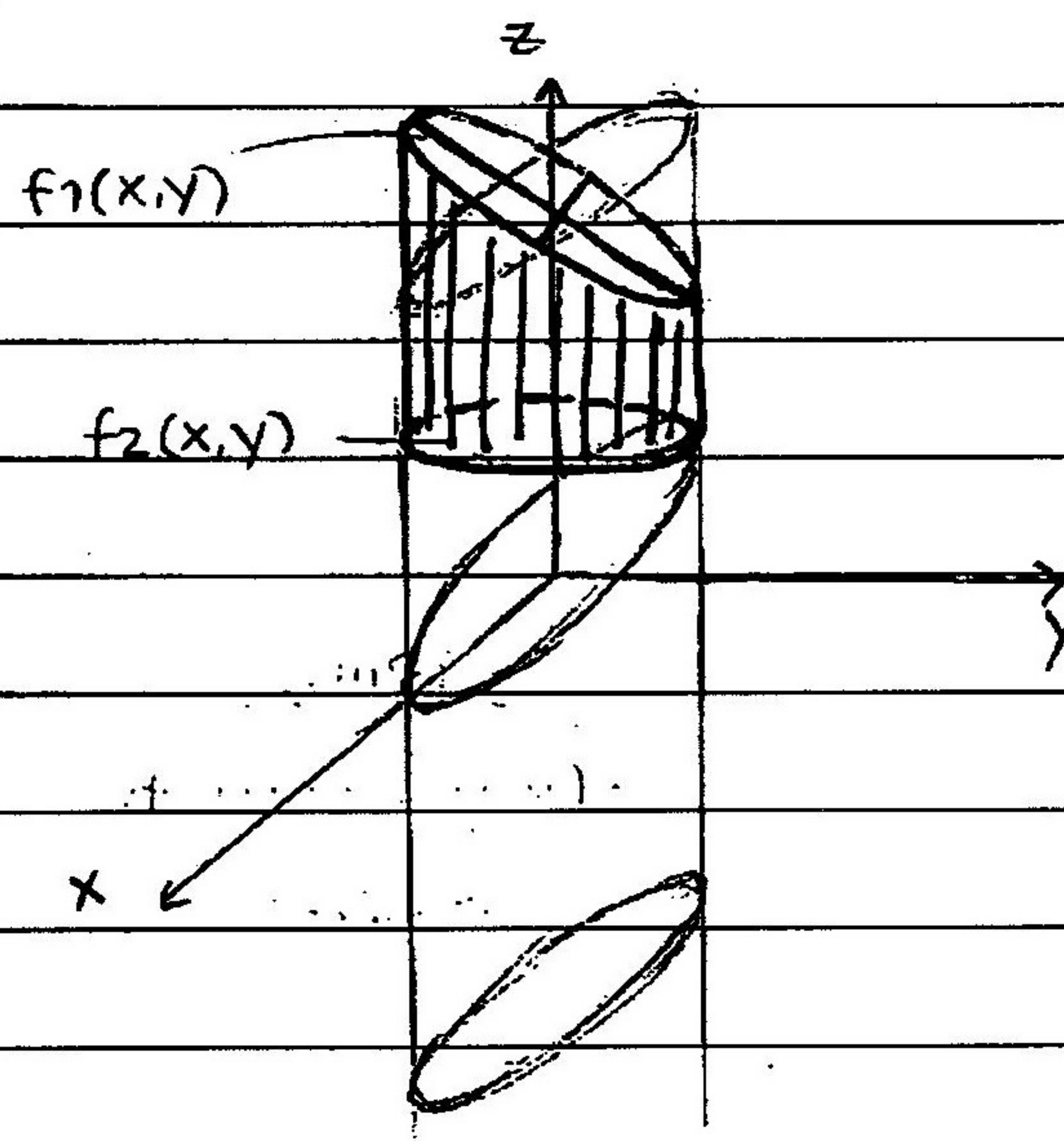
11(ii) $\int_0^a dx \int_0^{\sqrt{a^2-x^2}} 3x\, dy = \int_0^a 3x \left\{ \int_0^{\sqrt{a^2-x^2}} dy \right\} dx = 3 \int_0^a x\sqrt{a^2-x^2}\, dx =$



$= 3 \int_0^a \frac{1}{2} \sqrt{a^2-x^2}\, d(x^2) = -\frac{3}{2} \int_0^a \sqrt{a^2-x^2}\, d(a^2-x^2) =$

$= \frac{3}{2} \left[\frac{(a^2-x^2)^{1/2+1}}{3/2} \right]_0 = a^3$

15.(ii) $x^2 + 4y^2 = 4 \quad z = 1 \quad z = 12 - 3x - 4y$

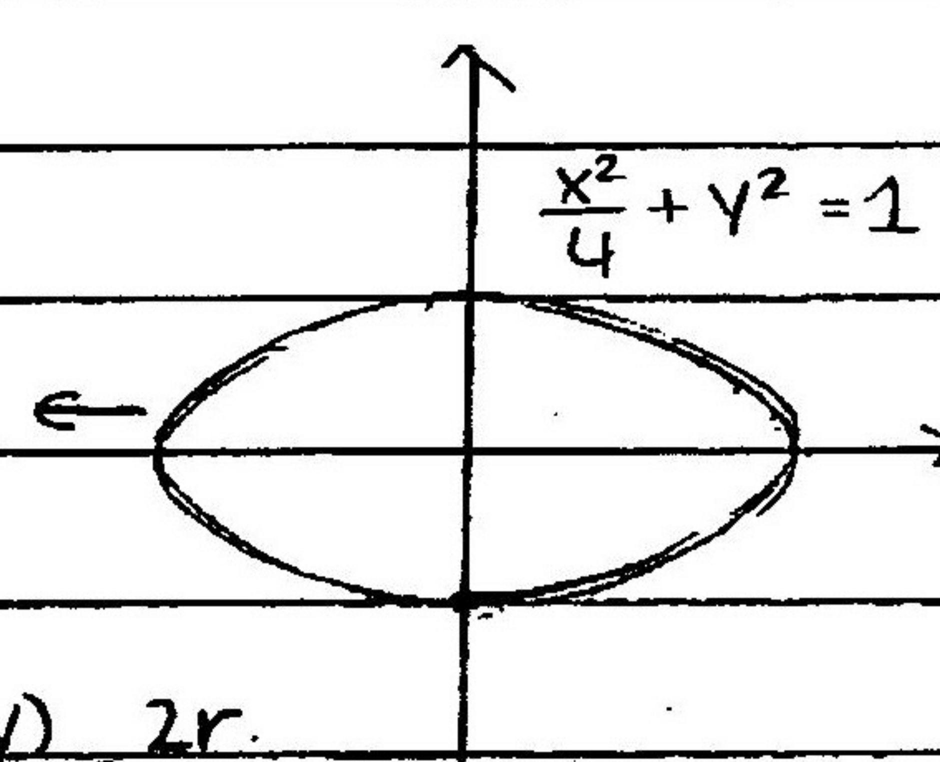


(όταν έχω τον όγκο ενός στερεού πρέπει να προσπαθήσω να βρω την προβολή T.)

$V = \iint_T f_1(x,y)\,dxdy - \iint_T f_2(x,y)\,dxdy$

$= \iint_T (11 - 3x - 4y)\,dxdy$

$= \iint_{D^*} (11 - 6r\cos\theta - 4r\sin\theta)\, 2r\,dr\,d\theta = \ldots = 22\pi$

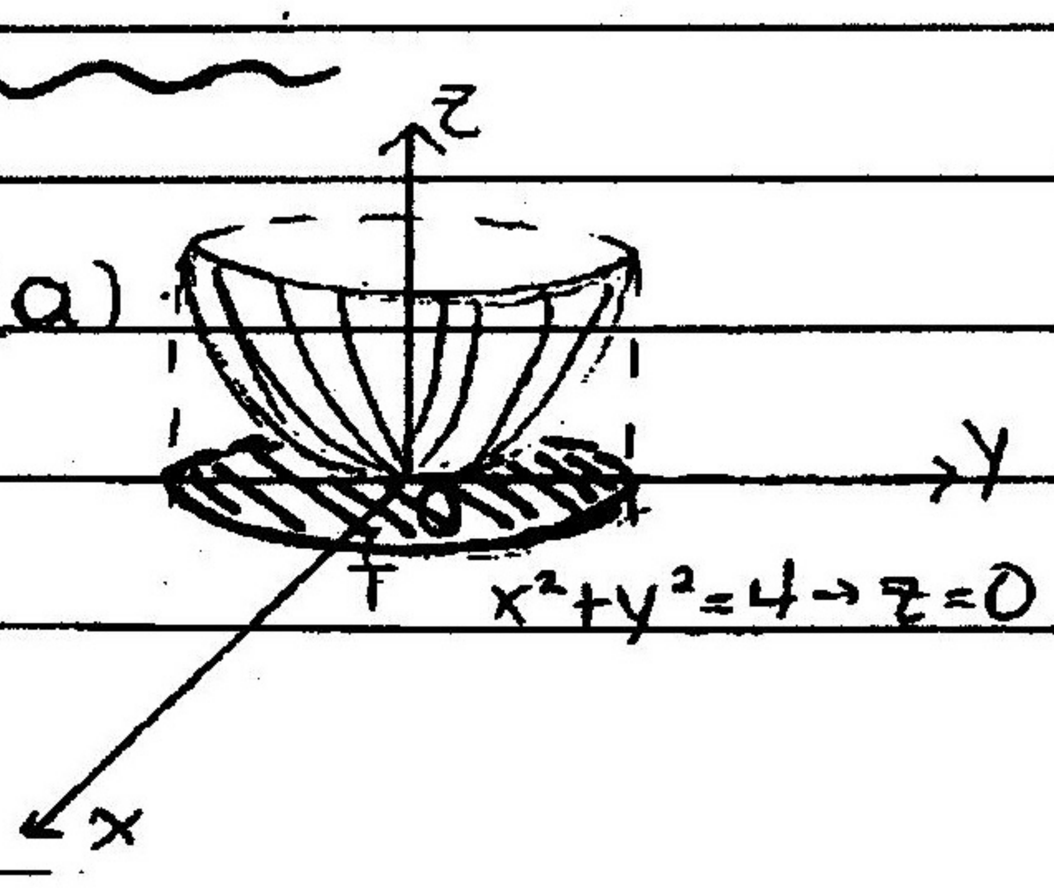


πολικές συντ.

$x = 2r\cos\theta$

$y = r\sin\theta \quad \frac{D(x,y)}{D(r,\theta)} = 2r$

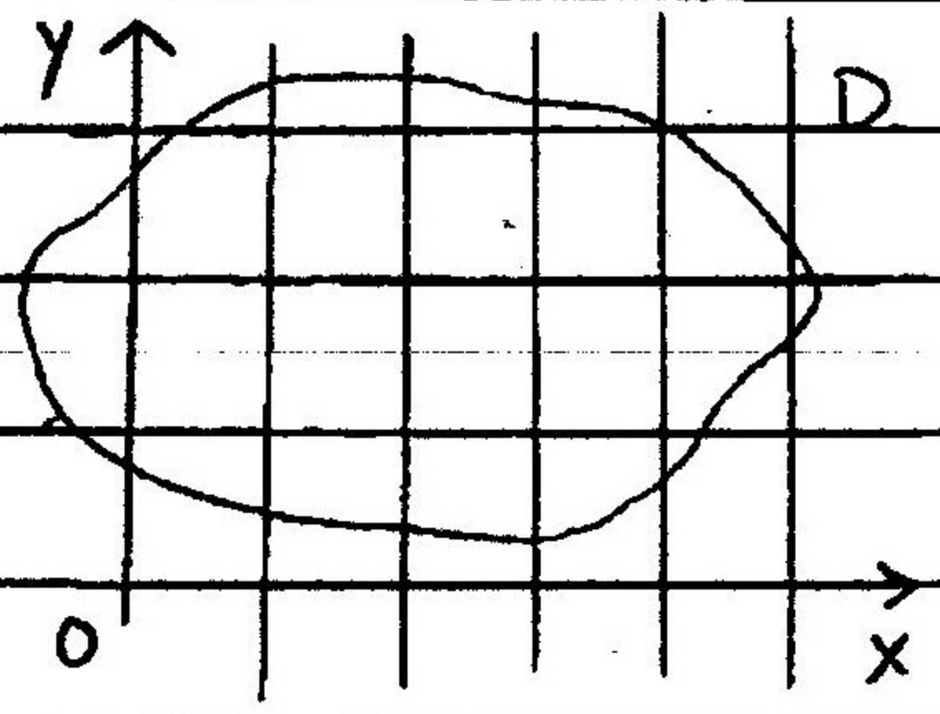
17.(a)

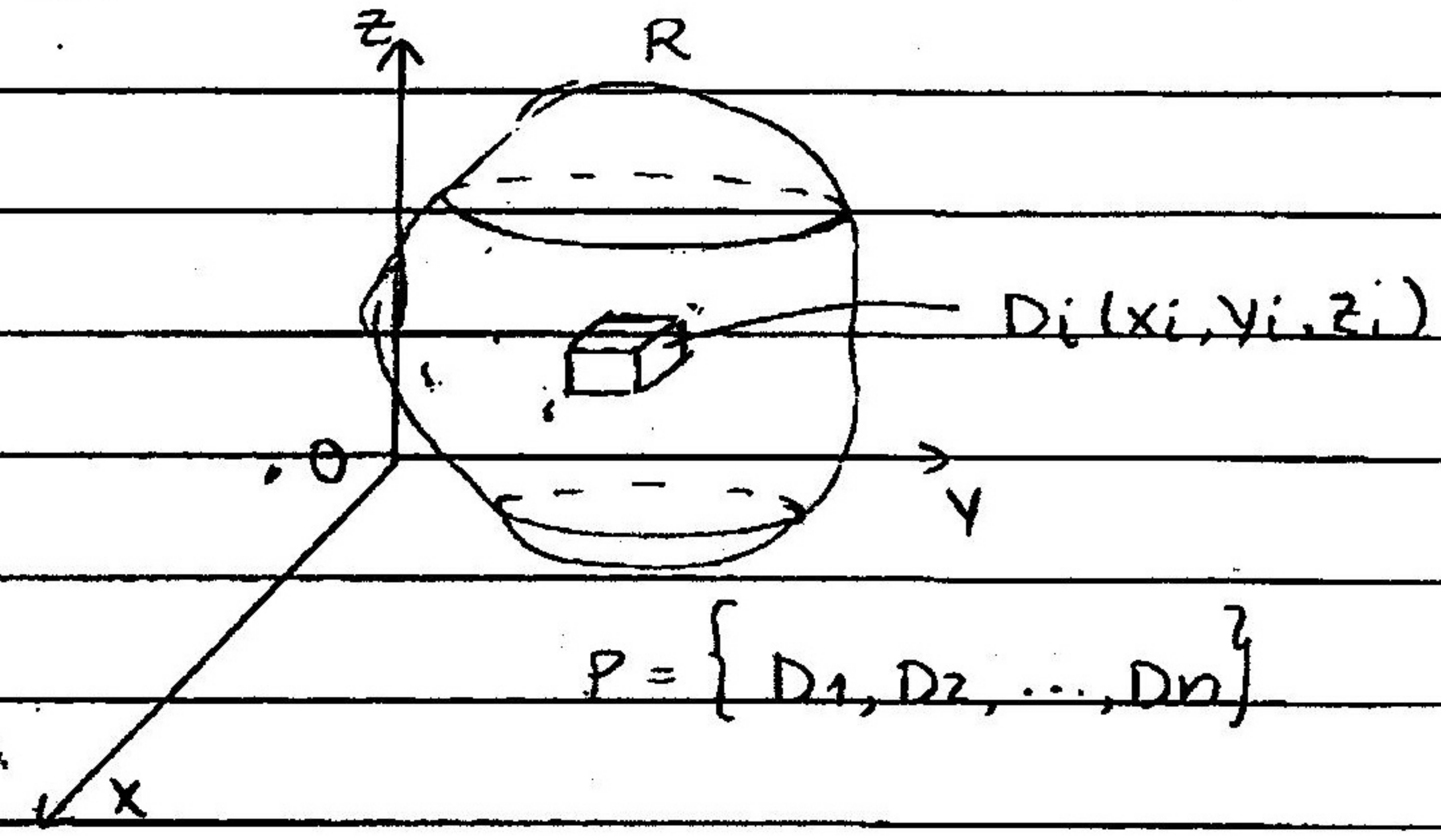


$\Omega: x^2 + y^2 = z \quad z = 4$

$V = \iint_T 4\,dxdy - \iint_T (x^2+y^2)\,dxdy$

ΤΡΙΠΛΑ ΟΛΟΚΛΗΡΩΜΑΤΑ





$w = f(x,y,z) / D \subset \mathbb{R}^3$, D = φραγμένο σύνολο, f = συνεχής και φραγμένο στο D.

διαμέριση του D: $P = \{D_1, D_2, \dots D_n\}$ $\forall i = 1, \dots, n$ $(x_i, y_i, z_i) \in D_i$

Άθροισμα Riemann:

$$S(f; P, \{(x_i, y_i, z_i)\}) = \sum_{i=1}^{n} f(x_i, y_i, z_i)\, V(D_i)$$

Ορισμός Τριπλού Ολοκληρώματος:

$w = f(x,y,z) / D \subset \mathbb{R}^3$, f = συνεχής και φραγμένο στο D, D = φραγμένο

τότε ονομάζουμε τριπλό ολοκλήρωμα:

$$\iiint_D f(x,y,z)\, dV = \lim_{\|P\| \to 0} \left[\sum_{i=1}^{n} f(x_i, y_i, z_i)\, V(D_i) \right]$$

Θεώρημα ύπαρξης:

- Αν η συνάρτηση $f(x,y,z)$ είναι συνεχής και φραγμένη στο $D \subset \mathbb{R}^3$ και το D είναι φραγμένο, τότε η f είναι ολοκληρώσιμη στο D.

Ιδιότητες:

① $f(x,y,z)$, $g(x,y,z)$ είναι ολοκληρώσιμες στο D και $\alpha, \beta \in \mathbb{R}^*$

τότε:

$$\iiint_D (\alpha f + \beta g)\, dV = \alpha \iiint_D f\, dV + \beta \iiint_D g\, dV.$$

② Αν $D_1, D_2 \subset \mathbb{R}^3$ $\quad \underbrace{V(D_1 \cap D_2)=0}$ και f = ολοκληρώσιμη στα D_1, D_2

είτε έχουμε 2 στερεά σώματα που δεν έχουν κοινά σημεία

ή τα κοινά σημεία δεν παράγουν όγκο.

↳ τότε:

$$\iiint_{D_1 \cup D_2} f\,dV = \iiint_{D_1} f\,dV + \iiint_{D_2} f\,dV$$

③ f, g - ολοκληρώσιμη στο D

$f(x,y,z) \leq g(x,y,z) \quad \forall (x,y,z) \in D$ τότε:

$$\iiint_D f\,dV \leq \iiint_D g\,dV$$

④ $f(x,y,z)$ = ολοκληρώσιμη ... φραγμένη στο D.

$m \leq f(x,y,z) \leq M \quad \forall (x,y,z) \in D$ τότε:

$$m V(D) \leq \iiint_D f\,dV \leq M V(D)$$

⑤ $f(x,y,z)$ = ολοκληρώσιμη στο D:

$$\left| \iiint_D f\,dV \right| \leq \iiint_D |f|\,dV$$

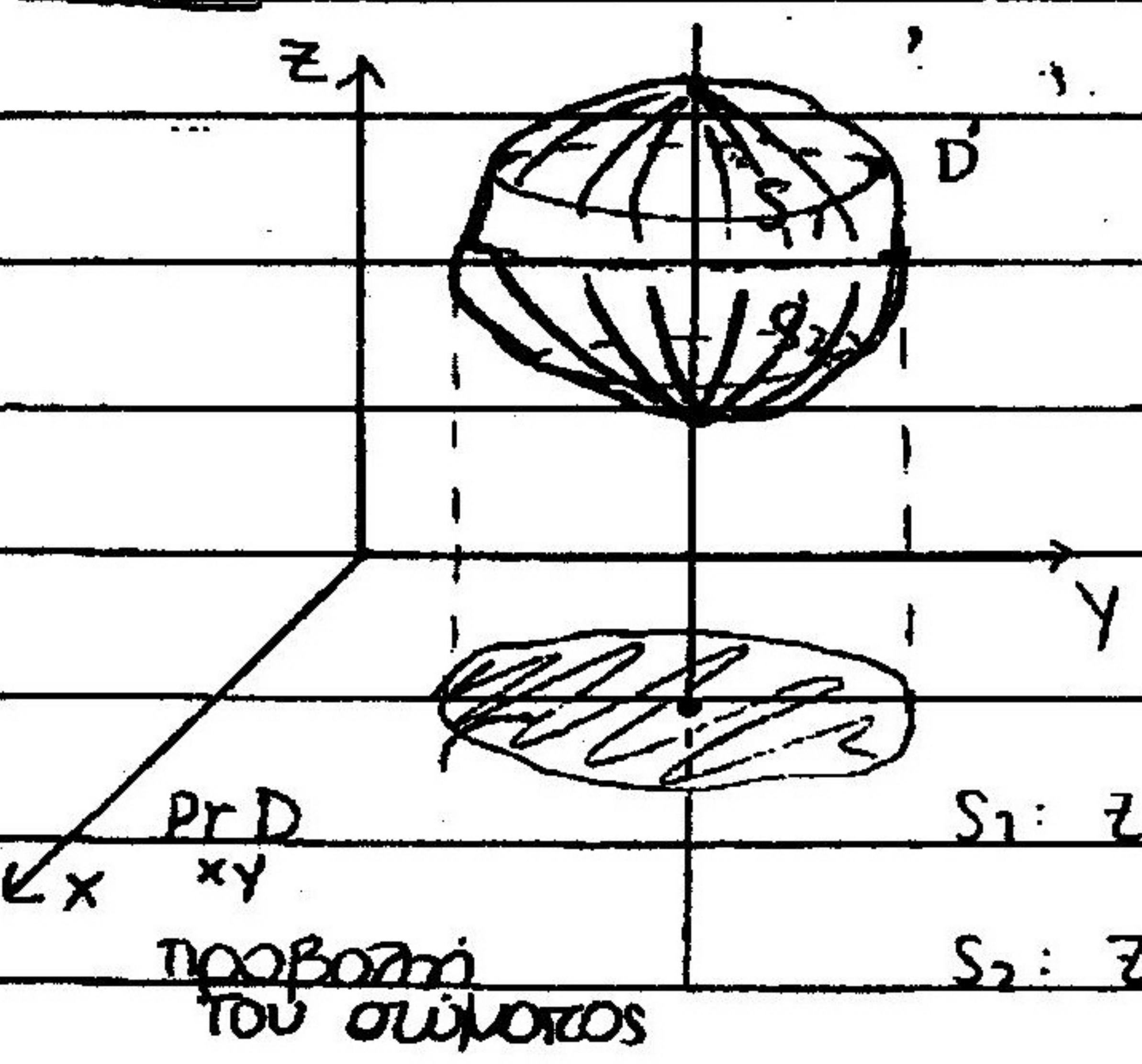
Θεώρημα Μέσης Τιμής:

Αν $w = f(x,y,z)$ → συνεχής και φραγμένη στο D και

D = κλειστό, φραγμένο + συνεχές $\Rightarrow \exists (\xi, \eta, \zeta) \in D$

$$f(\xi,\eta,\zeta) = \frac{1}{V(D)} \iiint_D f\,dV$$

Υπολογισμός Τριπλού Ολοκληρώματος



$$\iiint_D f(x,y,z)\,dV = \iint_{PrD_{xy}} \underbrace{\left\{ \int_{f_2(x,y)}^{f_1(x,y)} f\,dz \right\}}_{\Phi(x,y)} dx\,dy$$

$S_1: \ z = f_1(x,y)$

$S_2: \ z = f_2(x,y)$

για να βρω τι συμβαίνει στο z → πρέπει να βρω όλα τα ευθ.τμ. που έχουν αρχή στο πάνω μέρος, και τέλος στο κάτω.

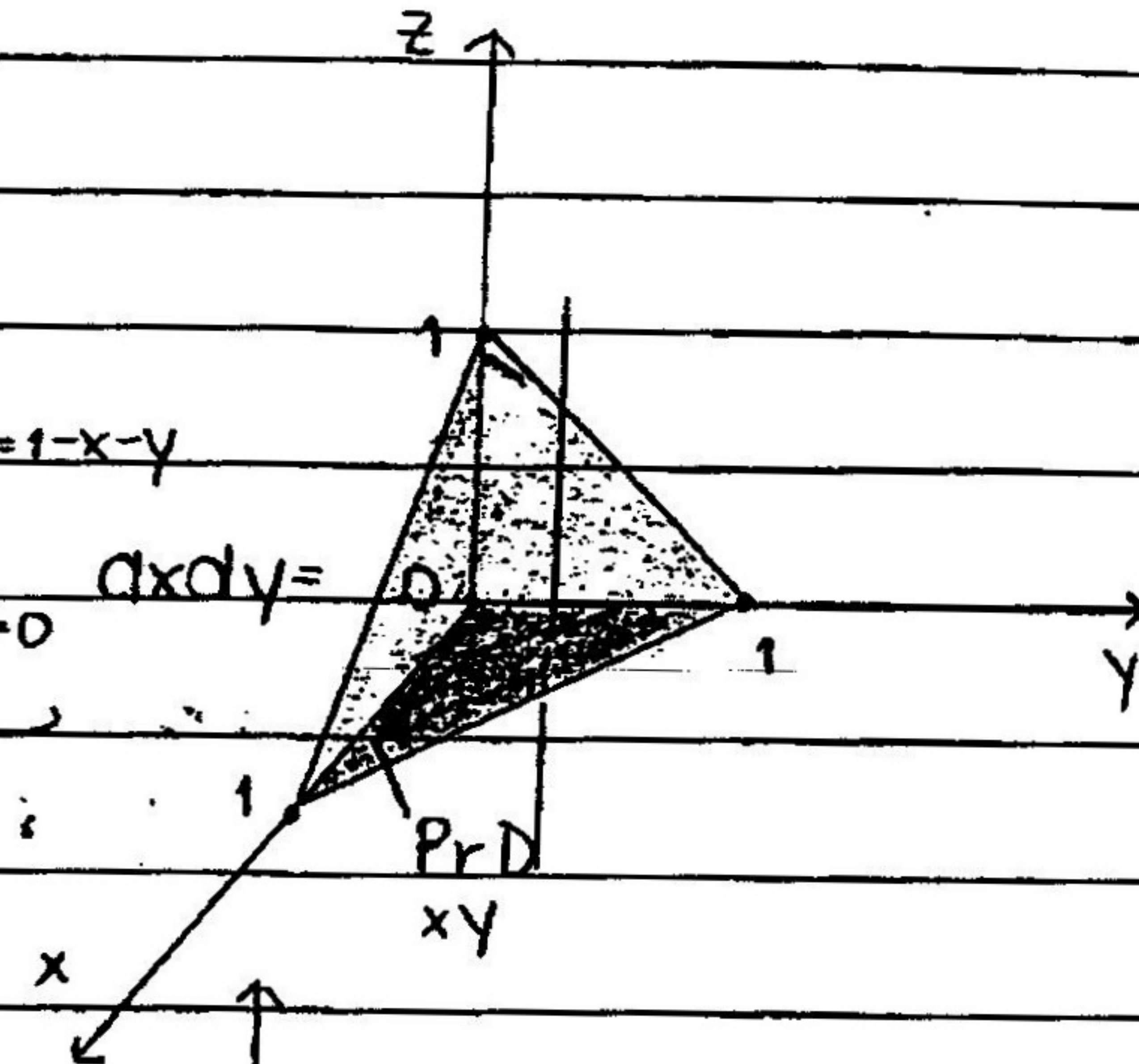
Π.χ.:

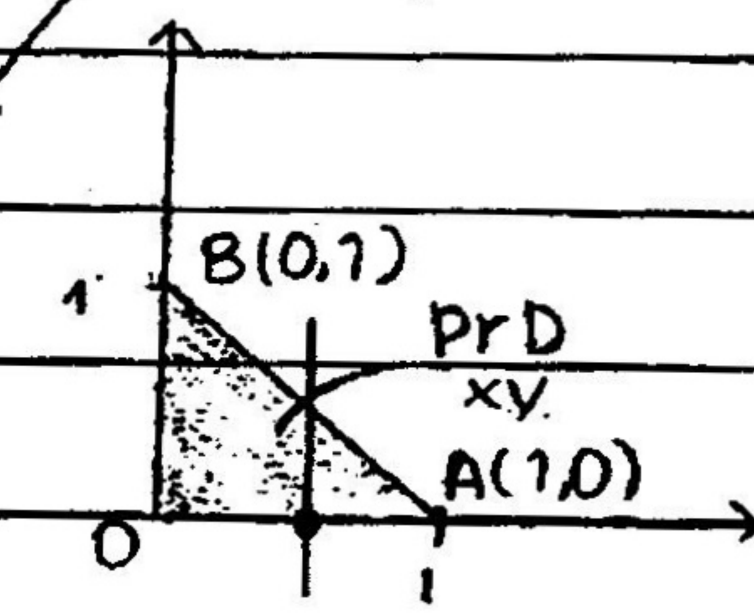
$$I = \iiint_D \frac{dx\,dy\,dz}{(1+x+y+z)^4} \qquad D: \begin{cases} x\geq 0,\ y \geq 0,\ z\geq 0 \\ x+y+z \leq 1 \end{cases}$$

$$I = \iint_{PrD_{xy}} \left\{ \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^4} \right\} dx\,dy = \iint_{PrD_{xy}} \underbrace{\left[-\frac{1}{3(1+x+y+z)^3} \right]_{z=0}^{z=1-x-y}}_{\Phi(x,y)} dx\,dy =$$

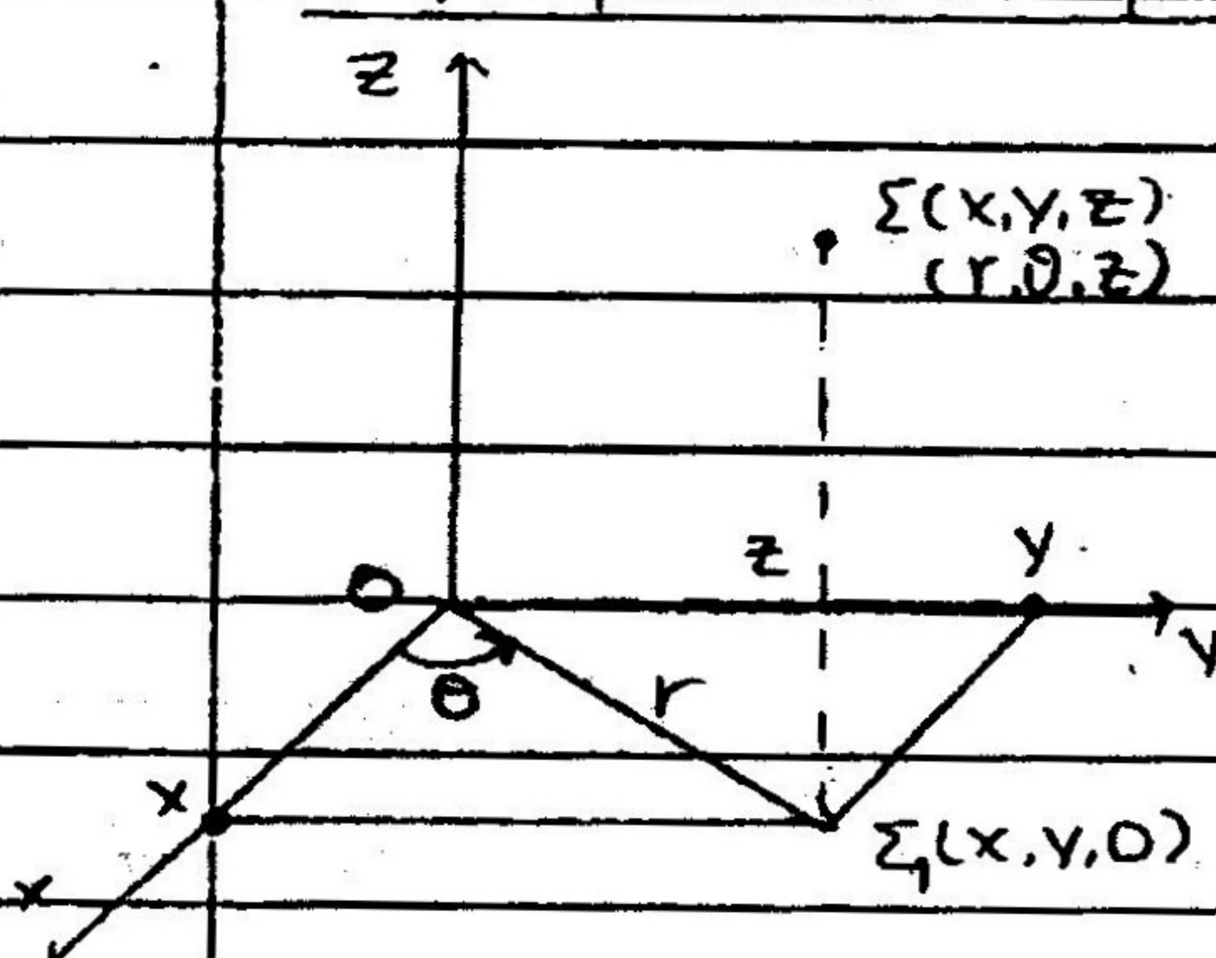
$$= -\frac{1}{3} \iint_{PrD_{xy}} \left(\frac{1}{8} - \frac{1}{3(1+x+y)^3} \right) dx\,dy$$

$$= -\frac{1}{3} \int_0^1 \left\{ \int_0^{1-x} \left(\frac{1}{8} - \frac{1}{(1+x+y)^3} \right) dy \right\} dx$$





Κυλινδρικό σύστημα συντεταγμένων.



κυλινδρικές συντεταγμένες

$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \theta = \arctan\frac{y}{x}$

$z = z \qquad z = z$

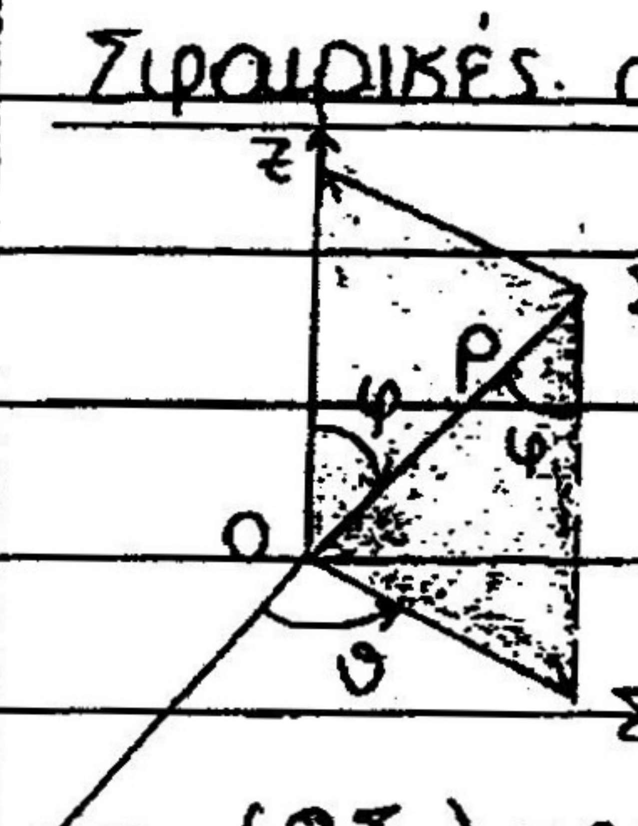
$z \subset \mathbb{R}$

$\theta \in [0, 2\pi)$

$r \in [0, +\infty)$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

Σφαιρικές συντεταγμένες



$(\rho,\theta,\varphi) \to$ σφαιρικές συντετ.

$\rho [0, +\infty) \qquad x = \rho\sin\varphi\cos\theta$

$\theta \in [0, 2\pi) \qquad y = \rho\sin\varphi\sin\theta$

$\varphi \in [0, \pi] \qquad z = \rho\cos\varphi$

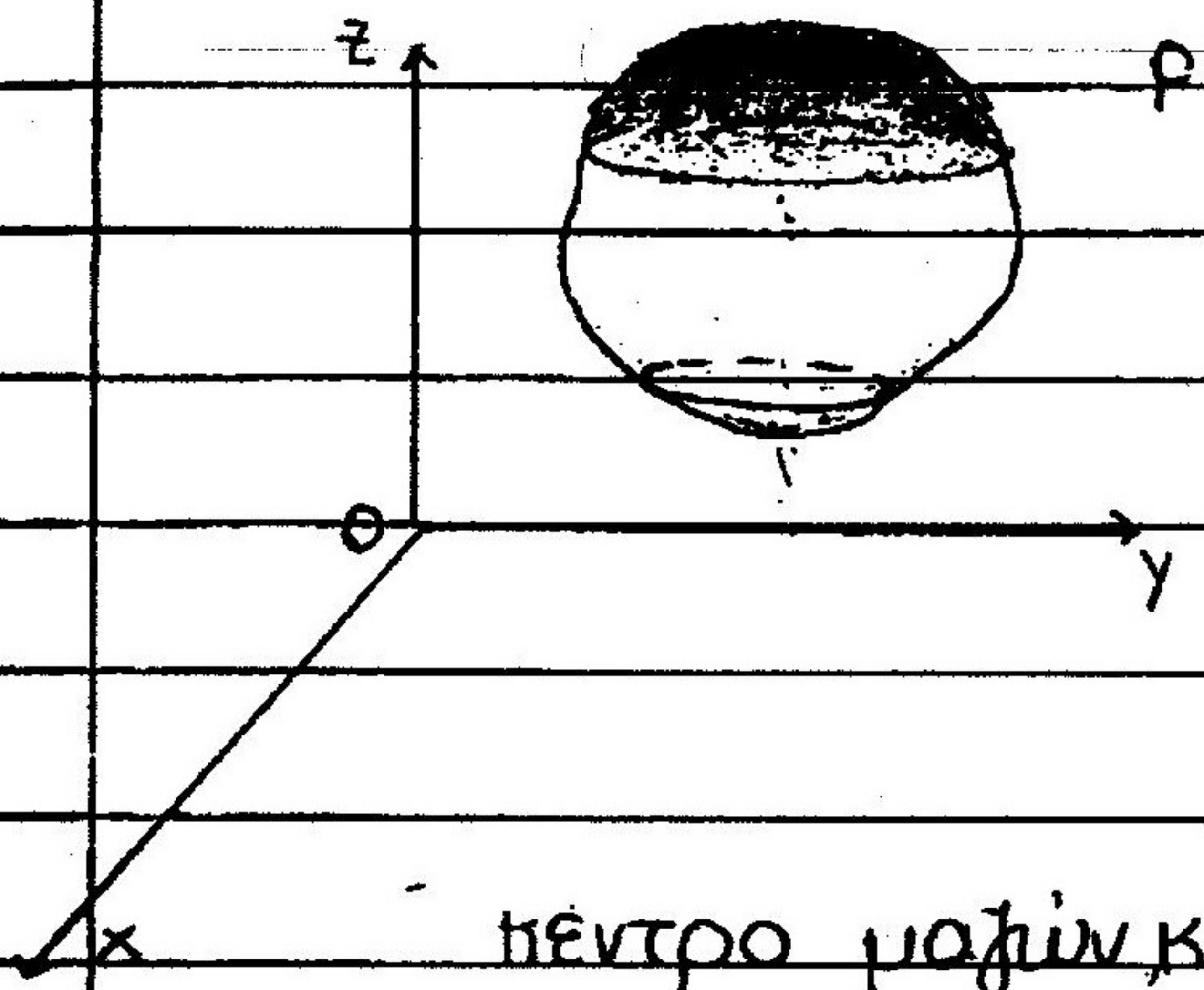
$(OΣ_1) = \rho\sin\varphi$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz \Rightarrow$$

$$\Rightarrow \iiint_{D^*} \underbrace{f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)}_{\Phi(\rho,\theta,\varphi)\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi} \overbrace{\frac{D(x,y,z)}{D(\rho,\theta,\varphi)}}^{\rho^2\sin\varphi} d\rho\, d\theta\, d\varphi$$

Εφαρμογές Τριπλών Ολοκληρωμάτων

① Μάζα στερεών σωμάτων



$\rho(x,y,z)$ $\qquad M = \iiint_D \rho \, dx\,dy\,dz$

$m_x = \iiint_D \rho x \, dx\,dy\,dz \qquad m_z = \iiint_D \rho z \, dx\,dy\,dz$

$m_y = \iiint_D \rho y \, dx\,dy\,dz$

κέντρο μαζών $K(x_K, y_K, z_K)$: $x_K = \dfrac{m_x}{M} \quad y_K = \dfrac{m_y}{M} \quad z_K = \dfrac{m_z}{M}$

ροπή αδράνειας:

$I_{xx} = \iiint_D (y^2+z^2)\rho \, dx\,dy\,dz \qquad I_{yy} = \iiint_D (x^2+z^2)\rho \, dx\,dy\,dz$

$I_{zz} = \iiint_D (x^2+y^2)\rho \, dx\,dy\,dz \qquad I_{xy} = \iiint_D xy\rho \, dx\,dy\,dz$

$$I = \begin{vmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{vmatrix}$$

Π.χ.:

- $I_1 = \int_0^1 \int_{x^2}^{x} \int_{x-y}^{x+y} y \, dz\,dy\,dx = \int_0^1 \int_{x^2}^{x} \left\{ \int_{x-y}^{x+y} y \, dz \right\} dy\,dx =$

$= \int_0^1 \int_{x^2}^{x} y \left[z \right]_{x-y}^{x+y} dy\,dx = \int_0^1 \int_{x^2}^{x} y(x+y-x+y)\,dy\,dx =$

$= 2\int_0^1 \left(\int_{x^2}^{x} y^2 dy \right) dx = 2\int_0^1 \left[\dfrac{y^3}{3} \right]_{x^2}^{x} dx = 2\int_0^1 \left(\dfrac{x^5}{3} - \dfrac{x^6}{3} \right) dx = \dfrac{1}{14}$

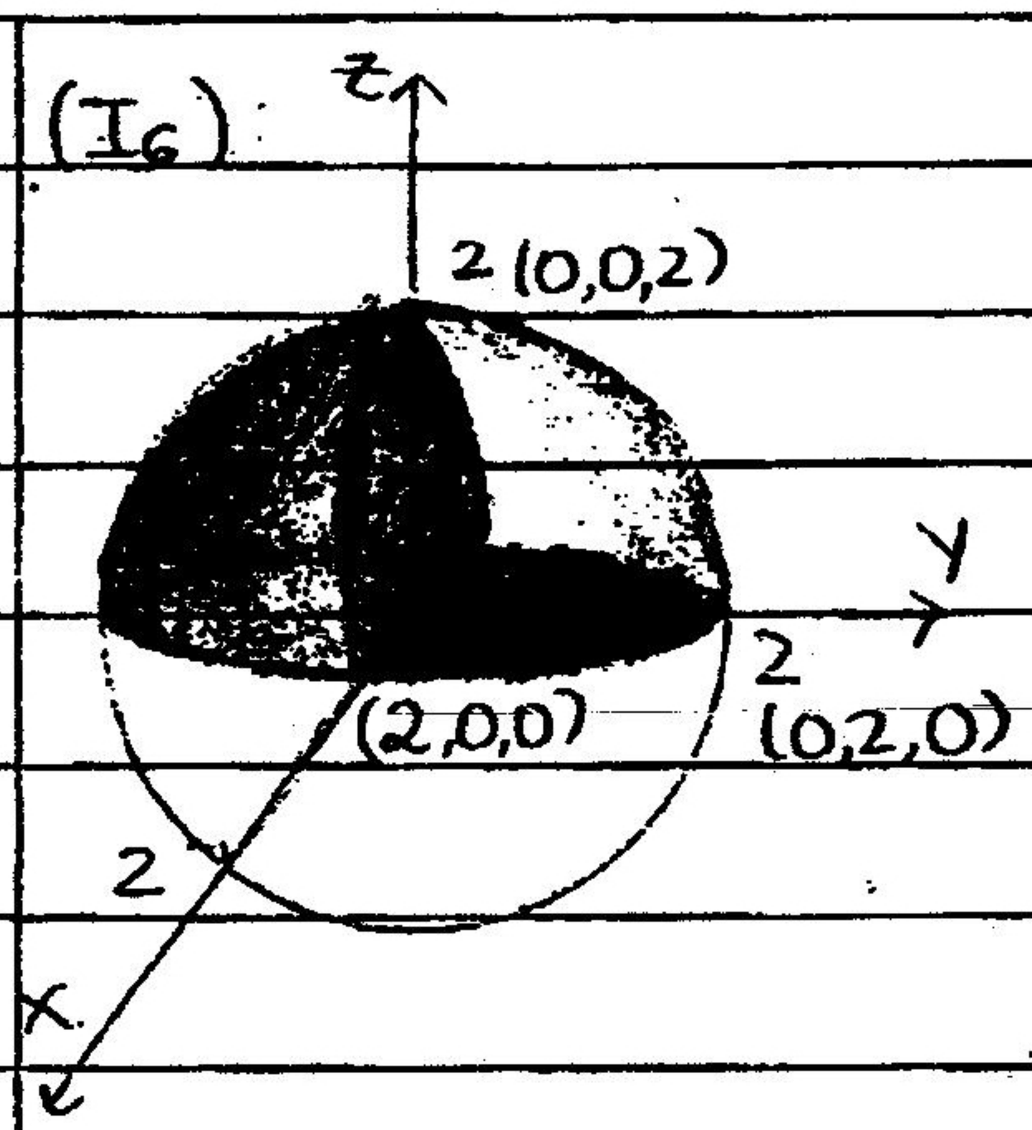
- $I_6 = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} z\sqrt{x^2+y^2+z^2} \, dz\,dy\,dx \qquad 0 \le y \le 2$

$-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$

$0 \le z \le \sqrt{4-x^2-y^2}$

→

(I_6)



$$(I_6) = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 1/2 \left[\frac{z^{3/2}}{3/2}\right]_{z=0}^{z=\sqrt{4-x^2-y^2}} dx\,dy.$$

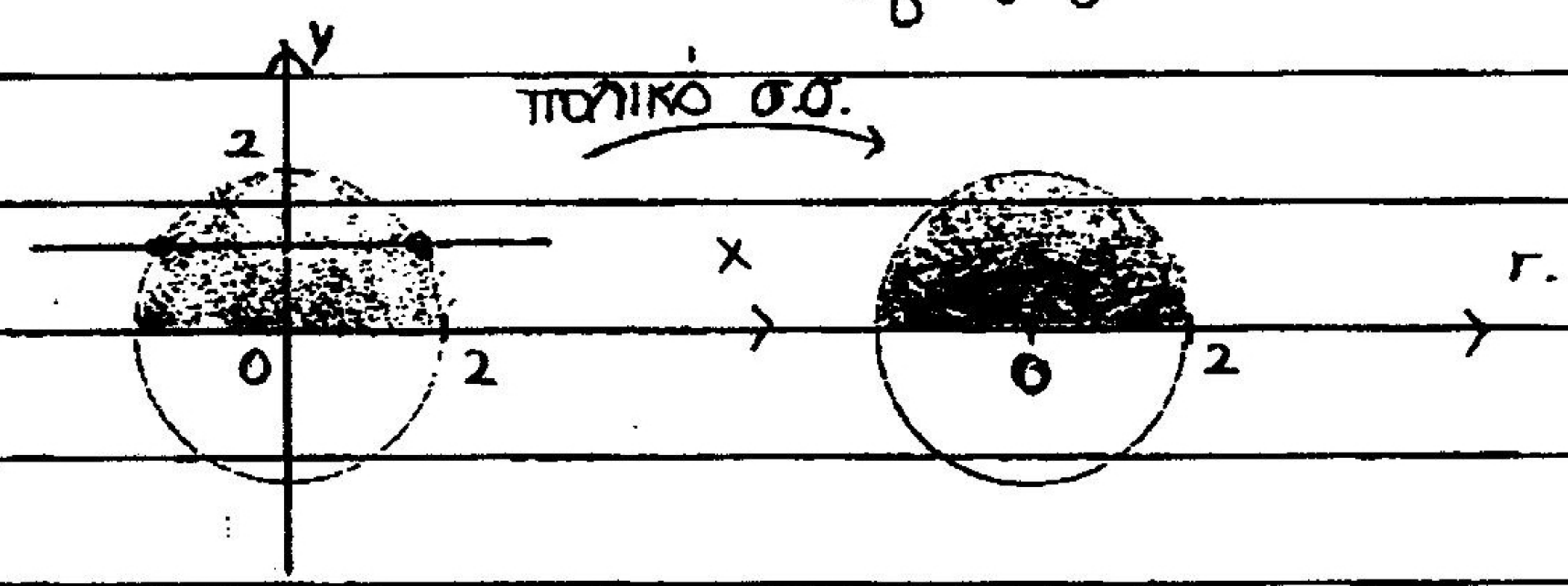
$$= 1/3 \int_0^2 \left\{ \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (\sqrt{4-x^2-y^2})^{3/2} dx \right\} dy =$$

*(αλλαγή σε πολικό σ.σ.)

$$= 1/3 \iint_{D^*} (\sqrt{4-r^2})^{3/2}\, r\,dr\,d\theta =$$

$$= \int_0^\pi \left\{ \int_0^2 (4-r^2)^{3/4}\, r\,dr \right\} d\theta = 16\pi/5.$$

*



- Σφαιρικό σύστημα συντεταγμένων. (I_6)

$\rho \in [0,2]$ $\quad x = \rho \sin\varphi \cos\theta$

$\theta \in [0,\pi]$ $\quad y = \rho \sin\varphi \sin\theta$

$\varphi \in [0,\pi/2]$ $\quad z = \rho \cos\varphi$

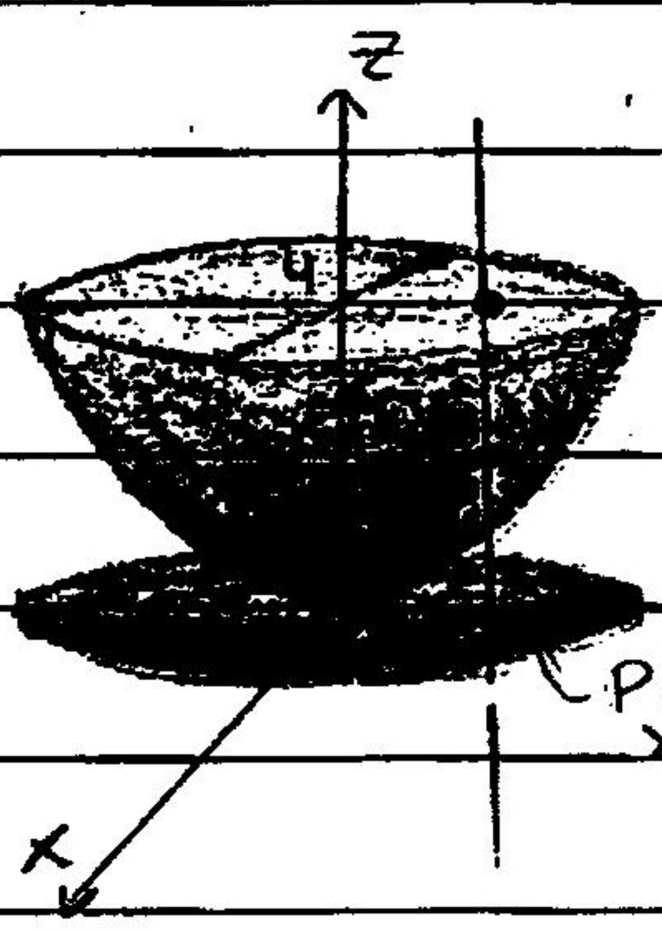
$$I_6 = \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{\pi} \left\{ \int_0^2 \rho\cos\varphi \cdot \rho^2 \sin\varphi \, d\rho \right\} d\theta\, d\varphi$$

ασκήσεις:

3(I) $z=4 \quad z = x^2+2y^2$

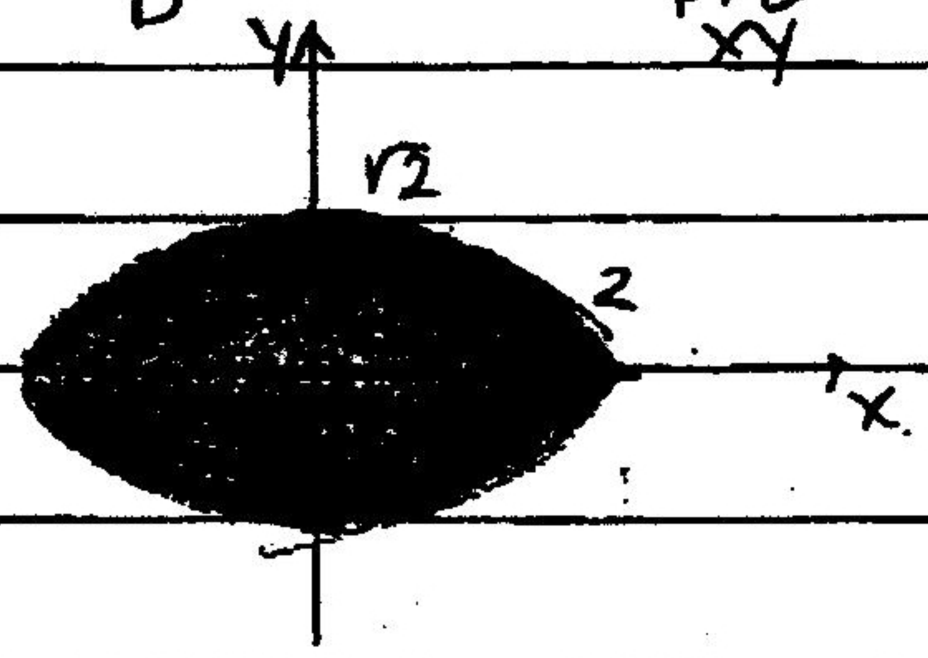
$x^2+2y^2 = 4 \quad z=4$

$\frac{x^2}{4} + \frac{y^2}{2} = 1 \quad z=4$



$Pr_{xy}D = \frac{x^2}{4} + \frac{y^2}{2} \le 1$

$$V = \iiint_D dx\,dy\,dz = \iint_{Pr_{xy}D} \left\{ \int_{x^2+2y^2}^{4} dz \right\} dx\,dy = \iint_{Pr_{xy}D} (4-x^2-2y^2)\, dx\,dy$$



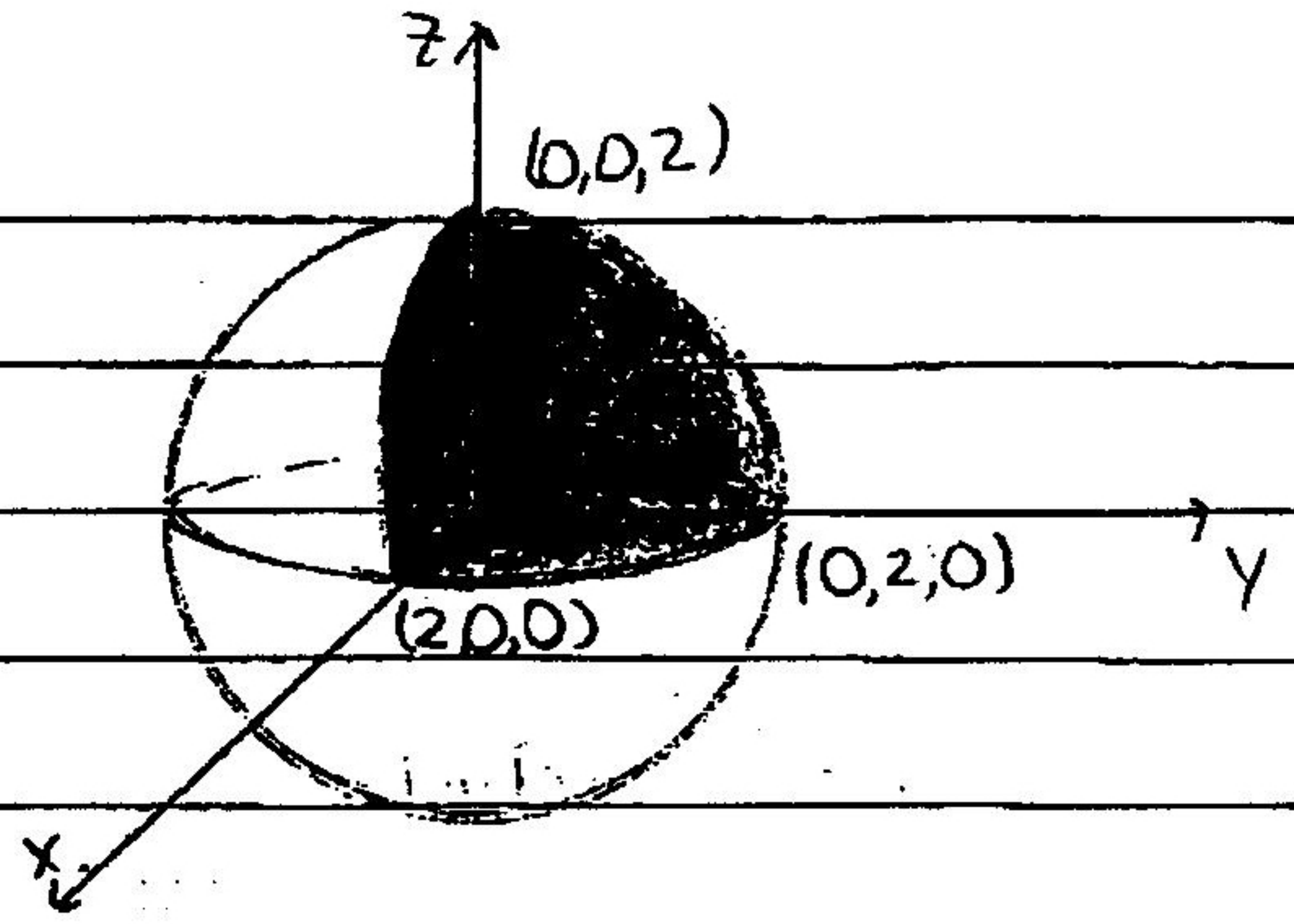
πολικές συντεταγμένες: $\quad \theta \in [0,2\pi]$

$x = 2r\cos\theta \quad r \in [0,1]$

$y = \sqrt{2}\, r\sin\theta \quad \frac{D(x,y)}{D(r,\theta)} = 2\sqrt{2}\, r.$

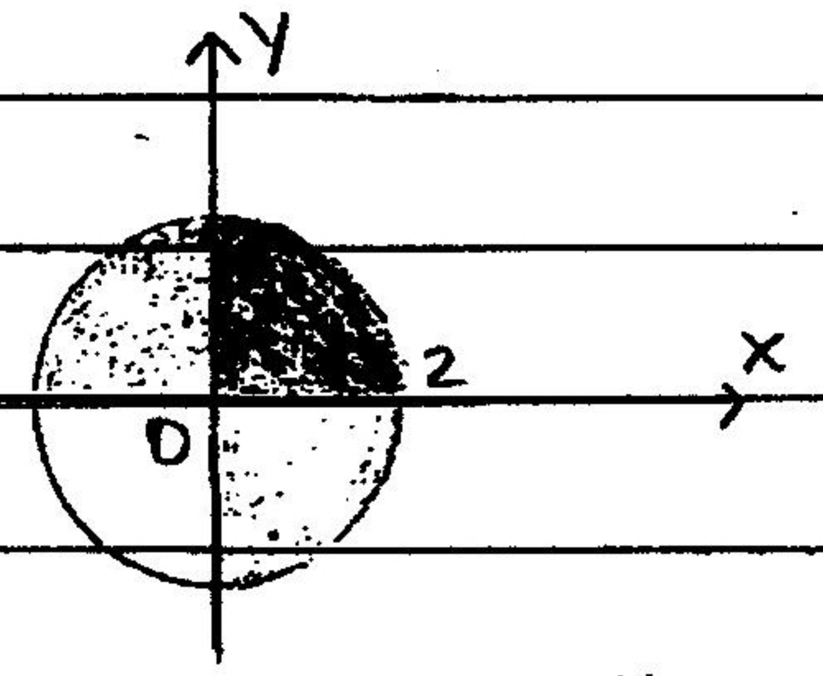
$$V = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (4 - r^2(\cos^2\theta \ldots) = 4\sqrt{2}\pi.$$

$$4(\Pi) \quad \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$$



$$-\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}$$

$$\left.\begin{array}{l} 0 \le y \le \sqrt{4-x^2} \\ 0 \le x \le 2 \end{array}\right]$$



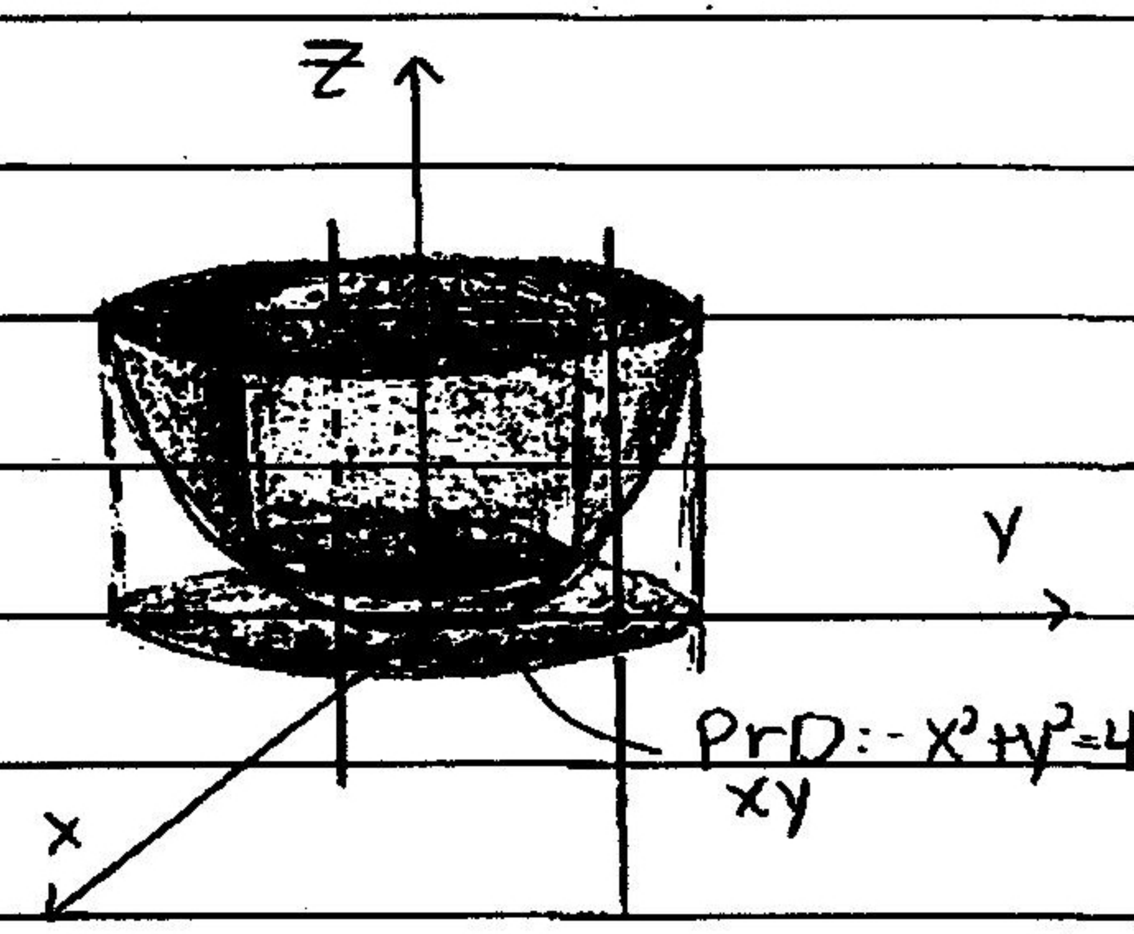
$$\rho \in [0,2] \quad \theta \in [0,\tfrac{\pi}{2}] \quad \varphi \in [0,\pi]$$

$$V = \underbrace{\int_{\rho=0}^{2} \int_{\varphi=0}^{\pi} \int_{\theta=0}^{\pi/2}}_{\rho,\varphi,\theta \to \text{ανεξάρτητες}} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi = \frac{8\pi}{3}$$

$5(\Pi)$ $z = x^2+y^2$ (κυκλικό παραβολοειδές)

$z=1$, $z=4$

$$V = \iint_{\substack{PrD \\ xy}} \left\{ \int_1^4 dz \right\} dx\,dy + \iint_{\substack{PrD \\ xy}\ \text{πολ}} \left\{ \int_{x^2+y^2}^{4} dz \right\} dx\,dy =$$



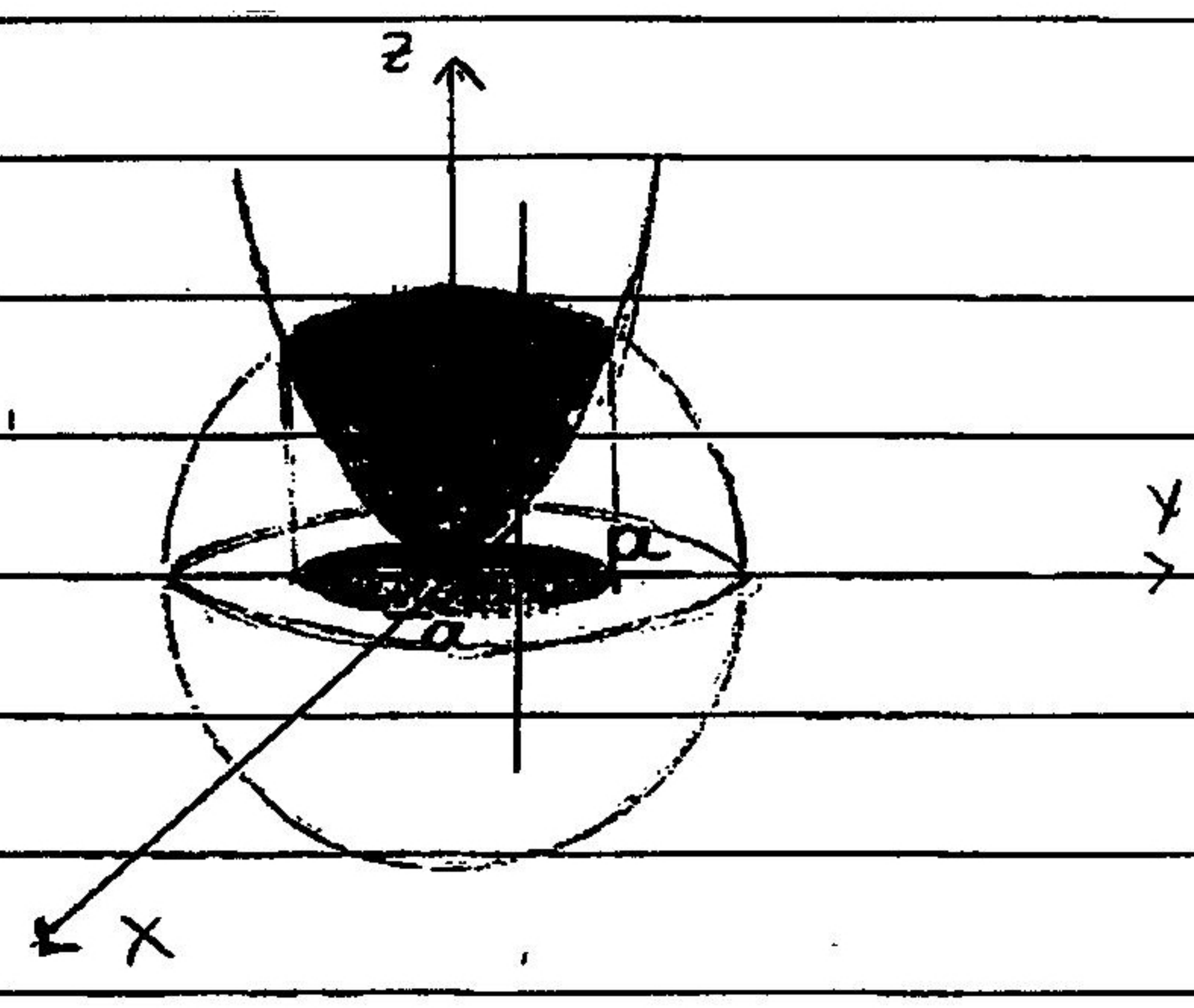
$$= 3 \iint_{\substack{\text{κύκλος} \\ x^2+y^2=1}} dx\,dy + \iint_{1 \le x^2+y^2 \le 4} (4-x^2-y^2)\,dx\,dy = \frac{15\pi}{2}$$

$5(V)$ $x^2+y^2+z^2 = 16$

$z = x^2+y^2$

$z^2+z-16=0$

$$z = \frac{-1 \pm \sqrt{1+64}}{2} = \frac{-1 \pm \sqrt{65}}{2}$$



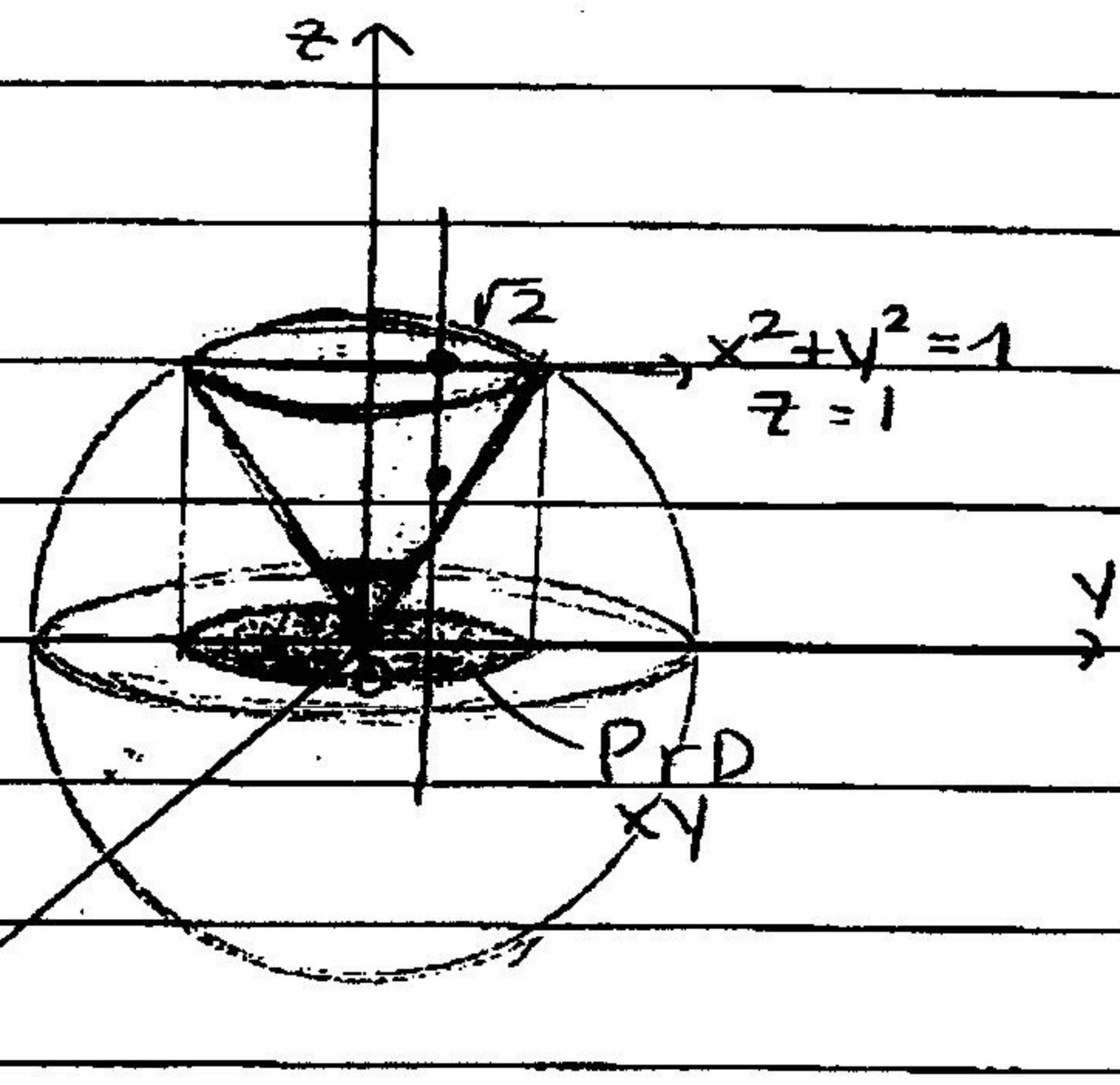
$$x^2+y^2 = a^2 = 16\left(\frac{-1 \pm \sqrt{65}}{2}\right)^2$$

$$V = \iiint_D dx\,dy\,dz = \iint_{\substack{PrD \\ xy}} \left\{ \int_{x^2+y^2}^{\sqrt{16-x^2-y^2}} dz \right\} dx\,dy = \dots = \frac{2\pi}{3}(6\sqrt{6}-11)$$

$x^2+y^2 \le a^2$

5(VII) $z=\sqrt{x^2+y^2}$ (κώνος)

$z=\sqrt{2-x^2-y^2}$ (σφαίρα)



$$V=\iiint_D dxdydz=\iint_{\substack{PrD\\xy}}\left\{\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}dz\right\}dxdy$$

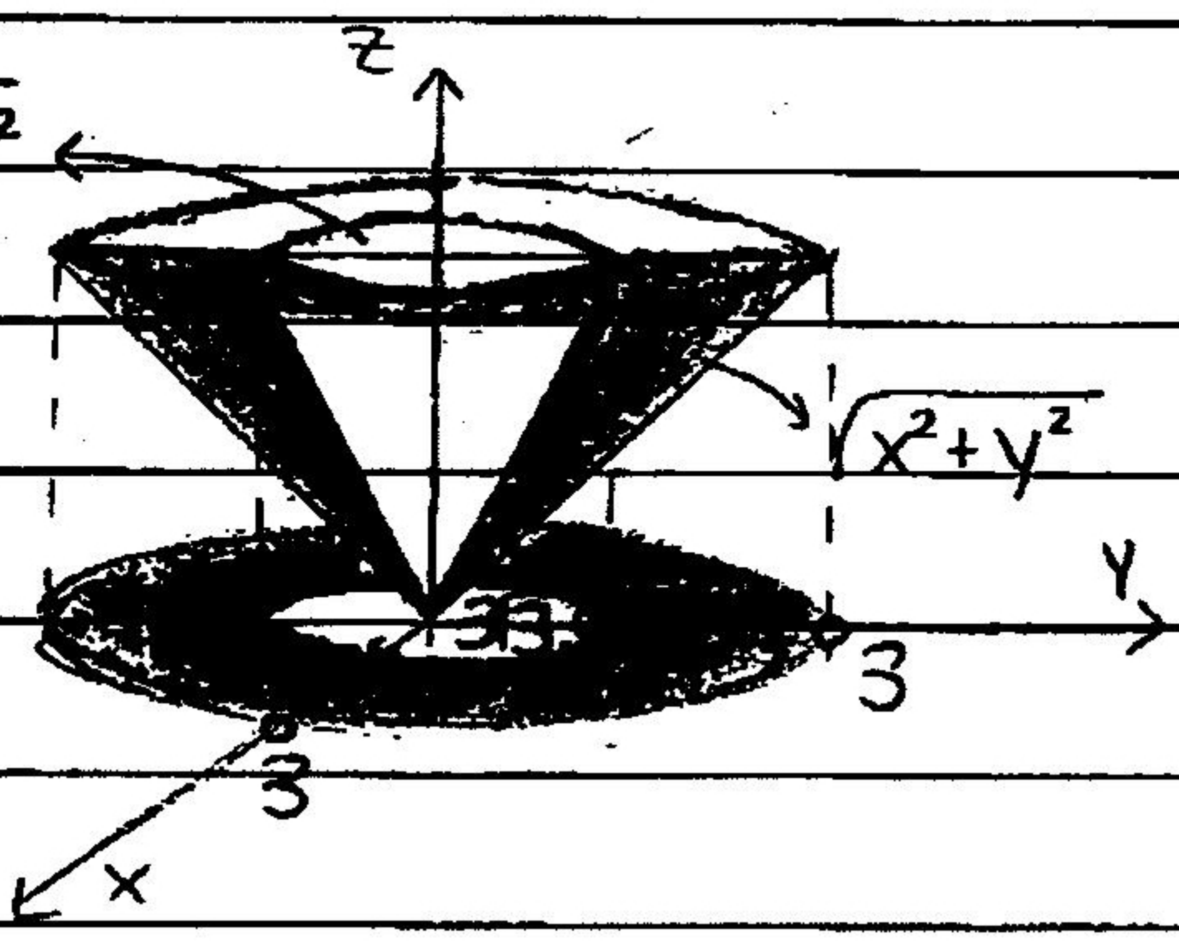
$$=\iint_{\substack{PrD\\xy}}\left(\sqrt{2-x^2-y^2}-\sqrt{x^2+y^2}\right)dxdy=$$

(αλλαγή σε πολικό σ.σ.)

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left(\sqrt{2-r^2}-r\right)r\,dr\,d\theta=\ldots=\frac{4\pi}{3}\left(\sqrt{2}-1\right)$$

6(δ) $z=\sqrt{x^2+y^2}$, $z=9$ $\rightarrow$ $x^2+y^2=9^2$

$z=9$, $z=\sqrt{3x^2+3y^2}$ $\rightarrow$ $3x^2+3y^2=9^2$, $x^2+y^2=27$



$$V_{εσωτ.}=\iiint_{\text{εσωτ. κύκλος}}dxdydz=\iint_{\substack{PrD\\xy}}\left\{\int_{\sqrt{x^2+y^2}}^{9}dz\right\}dxdy$$

(αλλαγή σε πολ. σ.σ.)

$$=\iint_{\substack{PrD\\xy}}\left(9-\sqrt{x^2+y^2}\right)dxdy=\int_{r=0}^{9}\int_{\theta=0}^{2\pi}(9-r)\,r\,dr\,d\theta=\ldots=108(3+\sqrt{3})\pi.$$

14 $\rho=\dfrac{1}{1+a^2}$ $\quad M=\iiint_D \rho\,dxdydz$ $\quad M=\iiint_D \dfrac{1}{1+x^2+y^2+z^2}dxdydz$

σφαιρικές συντεταγμένες

$\rho\in[0,1]$

$\theta\in[0,2\pi]$

$\varphi\in[0,\pi]$

$$M=\int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\frac{1}{1+\rho^2}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi=\pi(4-\pi)$$